1 Data Generating Process

- $T = 2$, $N = 10000$
- DGP

$$
\begin{aligned}
A_{i1} &= 1\{\pi_1 Z_{i1} + \alpha_i + v_{i1} \geq 0\} \\
Y_{i1} &= 1\{\mu_1 A_{i1} + \alpha_i + e_{i1} \geq 0\} \\
A_{i2} &= 1\{\pi_{21} Y_{i1} + \pi_{22} A_{i1} + \pi_{23} Z_{i2} + \alpha_i + v_{i2} \geq 0\} \\
Y_{i2} &= 1\{\mu_{21} Y_{i1} + \mu_{22} A_{i2} + \alpha_i + e_{i2} \geq 0\}
\end{aligned}
$$

  - $(v_1, e_1, v_2, e_2, \alpha)$, mutually independent, jointly normally distributed, mean γ_1, variance γ_2
  - (Z_1, Z_2), mutually independent, Bernoulli with β_1 and β_2

2 Linear Programming

2.1 Defininining State Variables

- Introduce

$$
\begin{aligned}
y_2 &= g_2(y_1, a_2, s_2) \\
a_2 &= h_2(a_1, y_1, z_2, r_2) \\
y_1 &= g_1(a_1, s_1) \\
a_1 &= h_1(z_1, r_1)
\end{aligned}
$$

Here, r_1 enumerates the map $z_1 \in \{0, 1\} \mapsto a_1 \in \{0, 1\}$ by enumerating possible value of potential treatment $a_1(z_1)$: $(a_1(0), a_1(1))$. That is,

$$
r_1 = \beta(a_1(0), a_1(1))
$$

r_1	$a_1(0)$	$a_1(1)$
1	0	0
2	1	0
3	0	1
4	1	1

Table 1: r_1

s_1	$y_1(0)$	$y_1(1)$
1	0	0
2	1	0
3	0	1
4	1	1

Table 2: s_1 as possible states of map $y_1(a_1)$

where $\beta(\cdot) = bi2de(\cdot)$. Likewise, s_1 enumerates the map $a_1 \in \{0,1\} \mapsto y_1 \in \{0,1\}$, r_2 enumerates $(a_1, y_1, z_2) \mapsto a_2$, and s_2 enumerates $(y_1, a_2) \mapsto y_2$:

$$\begin{aligned} s_1 &= \beta(y_1(0), y_1(1)) \\ r_2 &= \beta(a_2(0,0,0), a_2(0,0,1), ..., a_2(1,1,1)) \\ s_2 &= \beta(y_2(0,0), y_2(0,1), ..., y_2(1,1)) \end{aligned}$$

Then the corresponding sets have cardinality:

$$\begin{aligned} |\mathcal{R}_1| &= 2^2 \\ |\mathcal{S}_1| &= 2^2 \\ |\mathcal{R}_2| &= 2^{2^3} = 2^8 \\ |\mathcal{S}_2| &= 2^{2^2} = 2^4 \end{aligned}$$

- Now define

$$q(r_1, s_1, r_2, s_2) \equiv \Pr[r_1, s_1, r_2, s_2]$$

in $\mathcal{Q}$ with $|\mathcal{Q}| = |\mathcal{R}_1| \times |\mathcal{S}_1| \times |\mathcal{R}_2| \times |\mathcal{S}_2| = 2^2 \times 2^2 \times 2^8 \times 2^4 = 65,536$.

- Tables

r_2	$a_2(000)$	$a_2(100)$	$a_2(010)$	$a_2(110)$	$a_2(001)$	$a_2(101)$	$a_2(011)$	$a_2(111)$
1	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0
3	0	1	0	0	0	0	0	0
4	1	1	0	0	0	0	0	0
5	0	0	1	0	0	0	0	0
6	1	0	1	0	0	0	0	0
7	0	1	1	0	0	0	0	0
8	1	1	1	0	0	0	0	0
$\vdots$								
256	1	1	1	1	1	1	1	1

Table 3: r_2 as possible states of map $a_2(a_1, y_1, z_2)$

s_2	$y_2(00)$	$y_2(10)$	$y_2(01)$	$y_2(11)$
1	0	0	0	0
2	1	0	0	0
3	0	1	0	0
4	1	1	0	0
5	0	0	1	0
6	1	0	1	0
7	0	1	1	0
8	1	1	1	0
$\vdots$				
16	1	1	1	1

Table 4: s_2 as possible states of map $y_2(y_1, a_2)$

2.2 Potential Outcomes

- Then, e.g.,

$$\begin{aligned}
&\Pr[Y_1(1)=1, Y_2(1,1)=1]\\
&=\Pr[\boldsymbol{Y}(1,1)=(1,1)]\\
&=\Pr[S_1, S_2 : g_1(1,S_1)=1, g_2(1,1,S_2)=1]\\
&=\sum_{(\beta,\beta'):y_1^1=y_2^{11}=1} \Pr[S_1=\beta(y_1^0,y_1^1), S_2=\beta'(y_2^{00},y_2^{01},y_2^{10},y_2^{11})]\\
&=\sum_{(r_1,r_2)\in\mathcal{R}_1\times\mathcal{R}_2}\ \sum_{(\beta,\beta'):y_1^1=y_2^{11}=1} q[r_1,\beta(y_1^0,y_1^1),r_2,\beta'(y_2^{00},y_2^{01},y_2^{10},y_2^{11})]
\end{aligned}$$

and

$$\begin{aligned}
&\Pr[Y_1(1)=0, Y_2(1,1)=1]\\
&=\Pr[\boldsymbol{Y}(1,1)=(0,1)]\\
&=\sum_{(\beta,\beta'):y_1^1=0,y_2^{01}=1} \Pr[S_1=\beta(y_1^0,y_1^1), S_2=\beta'(y_2^{00},y_2^{01},y_2^{10},y_2^{11})]
\end{aligned}$$

Likewise,

$$\begin{aligned}
&\Pr[Y_1(0)=1, Y_2(1,1)=1]\\
&=\Pr[\boldsymbol{Y}(0,1)=(1,1)]\\
&=\sum_{(\beta,\beta'):y_1^0=y_2^{10}=1} \Pr[S_1=\beta(y_1^0,y_1^1), S_2=\beta'(y_2^{00},y_2^{01},y_2^{10},y_2^{11})]
\end{aligned}$$

and

$$\begin{aligned}
&\Pr[Y_1(1)=1, Y_2(1,0)=1]\\
&=\Pr[\boldsymbol{Y}(1,0)=(1,1)]\\
&=\sum_{(\beta,\beta'):y_1^1=y_2^{10}=1} \Pr[S_1=\beta(y_1^0,y_1^1), S_2=\beta'(y_2^{00},y_2^{01},y_2^{10},y_2^{11})]
\end{aligned}$$

And so on...

Regime #	d_1	$d_2(1, d_1)$	$d_2(0, d_1)$
1	0	0	0
2	1	0	0
3	0	1	0
4	1	1	0
5	0	0	1
6	1	0	1
7	0	1	1
8	1	1	1

Table 5: Dynamic Regimes

- Table for regime labels

$$\begin{aligned}
1 :& P[\boldsymbol{Y}(0,0) = (1,1)] + P[\boldsymbol{Y}(0,0) = (0,1)],\\
2 :& P[\boldsymbol{Y}(1,0) = (1,1)] + P[\boldsymbol{Y}(1,0) = (0,1)],\\
3 :& P[\boldsymbol{Y}(0,1) = (1,1)] + P[\boldsymbol{Y}(0,0) = (0,1)],\\
4 :& P[\boldsymbol{Y}(1,1) = (1,1)] + P[\boldsymbol{Y}(1,0) = (0,1)],\\
5 :& P[\boldsymbol{Y}(0,0) = (1,1)] + P[\boldsymbol{Y}(0,1) = (0,1)],\\
6 :& P[\boldsymbol{Y}(1,0) = (1,1)] + P[\boldsymbol{Y}(1,1) = (0,1)],\\
7 :& P[\boldsymbol{Y}(0,1) = (1,1)] + P[\boldsymbol{Y}(0,1) = (0,1)],\\
8 :& P[\boldsymbol{Y}(1,1) = (1,1)] + P[\boldsymbol{Y}(1,1) = (0,1)].
\end{aligned}$$

  - When $h_1 > 0$ and $h_2(o_1) > 0 \ \forall o_1$, then the analytical solution predicts that 7 and 8 are in the ID set

2.3 Constraints

- Number of constraints: $2^6 + 65,536 + 1$
  - but most of them are just defining the simplex
- Constraints: Let

$$p(a_1, y_1, a_2, y_2 | z_1, z_2)$$

be the distribution of the data

- Then, e.g.,

$$\begin{aligned}
p(0,0,0,0|0,0) &= \Pr[A_1(0)=0, Y_1(0)=0, A_2(0,0,0)=0, Y_2(0,0)=0 | Z_1=0, Z_2=0] \\
&= \Pr[A_1(0)=0, Y_1(0)=0, A_2(0,0,0)=0, Y_2(0,0)=0] \\
&= \sum_{(\beta,\beta',\beta'',\beta'''):a_1^0=y_1^0=a_2^{000}=y_2^{00}=0} q[\beta(a_1^0,a_1^1),\beta'(y_1^0,y_1^1),\beta''(a_2^{000},a_2^{001},...,a_2^{111}),\beta'''(y_2^{00},y_2^{01},y_2^{10},y_2^{11})]
\end{aligned}$$

and

$$\begin{aligned}
p(1,1,1,1|1,1) &= \sum_{(\beta,\beta',\beta'',\beta'''):a_1^1=y_1^1=a_2^{111}=y_2^{11}=1} q[\beta(a_1^0,a_1^1),\beta'(y_1^0,y_1^1),\beta''(a_2^{000},a_2^{001},...,a_2^{111}),\beta'''(y_2^{00},y_2^{01},y_2^{10},y_2^{11})] \\
&= \Pr[\boldsymbol{Y}(1,1)=(1,1), \boldsymbol{A}(1,1)=(1,1)]
\end{aligned}$$

Note that

$$\begin{aligned}
&\Pr[\boldsymbol{Y}(1,1)=(1,1)] \\
&= \Pr[\boldsymbol{Y}(1,1)=(1,1), A(1,1)=(0,0)] + \Pr[\boldsymbol{Y}(1,1)=(1,1), A(1,1)=(0,1)] \\
&+ \Pr[\boldsymbol{Y}(1,1)=(1,1), A(1,1)=(1,0)] + \Pr[\boldsymbol{Y}(1,1)=(1,1), A(1,1)=(1,1)] \\
&= \sum_{\beta:y_1^1=y_2^{11}=1} \sum_{\beta:a_1^1=a_2^{101}=0} q + \sum_{\beta:y_1^1=y_2^{11}=1} \sum_{\beta:a_1^1=0,a_2^{101}=1} q + \sum_{\beta:y_1^1=y_2^{11}=1} \sum_{\beta:a_1^1=1,a_2^{111}=0} q + \sum_{\beta:y_1^1=y_2^{11}=1} \sum_{\beta:a_1^1=a_2^{111}=1} q \\
&= \sum_{\beta:y_1^1=y_2^{11}=1} \sum_{(r_1,r_2)\in\mathcal{R}_1\times\mathcal{R}_2} q[r_1,\beta(y_1^0,y_1^1),r_2,\beta(y_2^{00},y_2^{01},y_2^{10},y_2^{11})]
\end{aligned}$$

and so on...

- Constraints 2 (Simplex):

$$\begin{aligned}
\sum_{r_1,s_1,r_2,s_2} q[r_1,s_1,r_2,s_2] &= 1 \\
q[r_1,s_1,r_2,s_2] &\geq 0 \quad \forall(r_1,s_1,r_2,s_2)
\end{aligned}$$

2.4 More Constraints

- Consider instead

$$
\begin{aligned}
y_2 &= g_2(y_1, a_2, s_2) \\
a_2 &= h_2(y_1, z_2, r_2) \\
y_1 &= g_1(a_1, s_1) \\
a_1 &= h_1(z_1, r_1)
\end{aligned}
$$

2.5 Population Distribution of Data

- Calculating $p(a_1, y_1, a_2, y_2 | z_1, z_2)$
- Recall

$$
\begin{aligned}
A_{i1} &= 1\{\pi_1 Z_{i1} + \alpha_i + v_{i1} \geq 0\} \\
Y_{i1} &= 1\{\mu_1 A_{i1} + \alpha_i + e_{i1} \geq 0\} \\
A_{i2} &= 1\{\pi_{21} Y_{i1} + \pi_{22} A_{i1} + \pi_{23} Z_{i2} + \alpha_i + v_{i2} \geq 0\} \\
Y_{i2} &= 1\{\mu_{21} Y_{i1} + \mu_{22} A_{i2} + \alpha_i + e_{i2} \geq 0\}
\end{aligned}
$$

- Then

$$
\begin{aligned}
p(0,0,0,0|0,0) &= \Pr[A_1(0) = 0, Y_1(0) = 0, A_2(0,0,0) = 0, Y_2(0,0) = 0 | Z_1 = 0, Z_2 = 0] \\
&= \Pr[A_1(0) = 0, Y_1(0) = 0, A_2(0,0,0) = 0, Y_2(0,0) = 0] \\
&= \Pr[\alpha_i + v_{i1} < 0, \alpha_i + e_{i1} < 0, \alpha_i + v_{i2} < 0, \alpha_i + e_{i2} < 0]
\end{aligned}
$$

and

$$
\begin{aligned}
p(1,1,1,1|1,1) &= \Pr[A_1(1) = 1, Y_1(1) = 1, A_2(1,1,1) = 1, Y_2(1,1) = 1] \\
&= \Pr[\pi_1 + \alpha_i + v_{i1} \geq 0, \mu_1 + \alpha_i + e_{i1} \geq 0, \pi_{21} + \pi_{22} + \pi_{23} + \alpha_i + v_{i2} \geq 0, \mu_{21} + \mu_{22} + \alpha_i + e_{i2} \geq 0
\end{aligned}
$$