Individualized Treatment Allocations with Distributional Welfare

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Policies for Heterogeneous Population

Individuals are heterogeneous

- so are their responses to treatments

When designing policies (i.e., treatment allocations), important to reflect this heterogeneity

$\Rightarrow$ individualized policies

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Policy design depends on policymaker's specific objective

- utilitarian (i.e., sum or mean) (Manski 04)

  vs.

- non-utilitarian (e.g., prioritarian, maximin)

Exploring Non-Utilitarian Paradigm

There may be settings where utilitarian goal is less sensible

- especially when target population exhibits skewed heterogeneity (e.g., outliers)
- possibility of non-utilitarian welfare (Manski 04)

The purpose of this paper: To explore objectives of (non-utilitarian) policymaker who concerns...

- distribution (e.g., tails) of treatment effects
- vote shares

Preliminaries

Observables:

- Y: outcome; D: binary treatment; X: covariates

Unobservables:

- Y_d (for $d = 1, 0$): potential outcomes

Policy:

- $\delta : \mathcal{X} \to \mathcal{A} \subseteq [0, 1]$ is a treatment allocation rule based on X
  - e.g., $\mathcal{A} = \{0, 1\}$ corresponds to the deterministic rule
  - e.g., $\mathcal{A} = [0, 1]$ corresponds to the stochastic rule
- $\mathcal{D}$: (potentially constrained) space of δ

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A policymaker (PM) wants to choose $\delta \in \mathcal{D}$ that optimizes a certain welfare criterion

Review: Mean (Utilitarian) Welfare

Utilitarian PM is interested in optimal policy δ^*_{ATE} that satisfies

$$\delta^*_{ATE} \in \arg\max_{\delta \in \mathcal{D}} E[\delta(X)Y_1 + (1 - \delta(X))Y_0]$$

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Because

$$\begin{aligned} E[\delta(X)Y_1 + (1-\delta(X))Y_0] &= E[Y_0 + \delta(X)(Y_1 - Y_0)] \\ &= E[Y_0] + E[\delta(X)E[Y_1 - Y_0|X]], \end{aligned}$$

$$\delta^*_{ATE} \in \arg\max_{\delta \in \mathcal{D}} E[\delta(X)E[Y_1 - Y_0|X]]$$

- conditional ATE as "welfare gain"
- subject to constraints, δ^*_{ATE} maximizes the average of conditional ATE selected (or weighted) by δ

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- with no constraint, $\delta^*_{ATE}(x) = 1\{E[Y_1 - Y_0 | X = x] \geq 0\}$

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- e.g., given $X = x$, few individuals with high $Y_1 - Y_0$ can make $E[Y_1 - Y_0 | X = x] > 0$

$Y_1 - Y_0 \mid X = x$
7
5
-1
-1
-1
-1
-1
-2
-2
-2

$$E[Y_1 - Y_0 | X = x] = 1/10$$

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- can be sensitive to outliers
- e.g., given $X = x$, few individuals with high $Y_1 - Y_0$ can make $E[Y_1 - Y_0 | X = x] > 0$
- suggests to treat *all* individuals with $X = x$ even though the treatment harms the majority

$Y_1 - Y_0 \| X = x$
7
5
-1
-1
-1
-1
-1
-2
-2
-2

$$E[Y_1 - Y_0 | X = x] = 1/10$$

This Paper: Quantile of Treatment Effects as Welfare Gain

We propose

$$\delta^* \equiv \delta^*_{QoTE} \in \arg\max_{\delta \in \mathcal{D}} E[\delta(X) Q_\tau(Y_1 - Y_0 | X)]$$

- $Q_\tau(Y_1 - Y_0|X)$ is τ-quantile of $Y_1 - Y_0$ (QoTE) given X

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- δ^* maximizes the average of conditional QoTE selected (or weighted) by δ
- with no constraint, $\delta^*(x) = 1\{Q_\tau(Y_1 - Y_0|X = x) \geq 0\}$
- τ (i.e., rank in individual TEs) represents a reference group chosen by the PM

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$$\delta^* \in \arg\max_{\delta \in \mathcal{D}} E[\delta(X) Q_\tau(Y_1 - Y_0|X)]$$

- decision less sensitive to outliers
  - "within-group fairness"
    (Leqi & Kennedy 21)

$Y_1 - Y_0|X = x$

7
5
-1
-1
-1
-1
-1
-2
-2
-2

$$Q_{0.5}(Y_1 - Y_0|X = x) = -1$$

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$Y_1 - Y_0 | X = x$

7
5
-1
-1
-1
-1
-1
-2
-2
-2

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- decision less sensitive to outliers
  - "within-group fairness" (Leqi & Kennedy 21)
- τ is chosen by the PM to set a reference group
  - large τ: negligent PM

$Y_1 - Y_0 | X = x$

7
5
-1
-1
-1
-1
-1
-2
-2
-2

$$Q_{0.9}(Y_1 - Y_0 | X = x) = 5$$

This Paper: Quantile of Treatment Effects as Welfare Gain

$$\delta^* \in \arg\max_{\delta \in \mathcal{D}} E[\delta(X) Q_\tau(Y_1 - Y_0|X)]$$

- decision less sensitive to outliers
  - "within-group fairness" (Leqi & Kennedy 21)
- τ is chosen by the PM to set a reference group
  - large τ: negligent PM
  - small τ: prudent PM

$Y_1 - Y_0|X = x$

7
5
−1
−1
−1
−1
−1
−2
−2
−2

$Q_{0.2}(Y_1 - Y_0|X = x) = -2$

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$Y_1 - Y_0 | X = x$

7
5
2
2
2
1
1
1
1
-2

$$Q_{0.2}(Y_1 - Y_0 | X = x) = 1$$

Alternatives in Literature: Quantile Welfare

Wang, Zhou, Song & Sherwood 18: Quantile of $Y_{\delta(X)}$, focusing on deterministic regime

$$\delta^*_? \in \arg\max_{\delta \in \mathcal{D}} Q_\tau(Y_{\delta(X)})$$

- no closed-form solution for optimal policy $\delta^*_?$
  - interpretation of welfare gain is elusive
- lack of "across-group fairness" (Leqi & Kennedy 21):
  - decision for one group is influenced by TEs of other groups

Alternatives in Literature: Quantile Welfare

Leqi & Kennedy 21: Average of conditional quantile, focusing on deterministic regime

$$\delta^*_{QTE} \in \arg\max_{\delta \in \mathcal{D}} E[Q_\tau(Y_{\delta(X)}|X)]$$

Alternatives in Literature: Quantile Welfare

Leqi & Kennedy 21: Average of conditional quantile, focusing on deterministic regime

$$\delta^*_{QTE} \in \arg\max_{\delta \in \mathcal{D}} E[Q_\tau(Y_{\delta(X)}|X)]$$

But because

$$\begin{aligned} E[Q_\tau(Y_{\delta(X)}|X)] &= E[\delta(X)Q_\tau(Y_1|X) + (1-\delta(X))Q_\tau(Y_0|X)] \\ &= E[Q_\tau(Y_0|X)] + E[\delta(X)\{Q_\tau(Y_1|X) - Q_\tau(Y_0|X)\}], \end{aligned}$$

$$\delta^*_{QTE} \in \arg\max_{\delta \in \mathcal{D}} E[\delta(X)\{Q_\tau(Y_1|X) - Q_\tau(Y_0|X)\}]$$

- δ^*_{QTE} maximizes the average of conditional QTE selected by δ
- with no constraint, $\delta^*_{QTE}(x) = 1\{Q_\tau(Y_1|X=x) - Q_\tau(Y_0|X=x) \geq 0\}$

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$$\delta^*_{QTE} \in \arg\max_{\delta \in \mathcal{D}} E[Q_\tau(Y_{\delta(X)}|X)]$$

- with no constraint, $\delta^*_{QTE}(x)$
  $= 1\{Q_\tau(Y_1|X = x)$
  $-Q_\tau(Y_0|X = x) \geq 0\}$

Alternatives in Literature: Quantile Welfare

$$\delta^*_{QTE} \in \arg\max_{\delta \in \mathcal{D}} E[Q_\tau(Y_{\delta(X)}|X)]$$

- with no constraint, $\delta^*_{QTE}(x) = 1\{Q_\tau(Y_1|X = x) - Q_\tau(Y_0|X = x) \geq 0\}$
- QTE is difference of Q_τ's of potentially different individuals
  - hard to justify esp. in making individualized decision
  - hard to define prudence or negligence

$Y_1 \mid X = x$	$Y_0 \mid X = x$
2	0
7	2
2	1
1	0
5	3
2	1
1	0
0	0
1	1
1	0

$$Q_{0.5}(Y_1) - Q_{0.5}(Y_0) = 1$$

This Paper: Quantile of Treatment Effects as Welfare Gain

$$\delta^* \in \arg\max_{\delta \in \mathcal{D}} E[\delta(X) Q_\tau(Y_1 - Y_0|X)]$$

- still, the notion of welfare *level* is unclear

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- still, the notion of welfare *level* is unclear

Another interpretation of $\delta^*(x) = 1\{Q_\tau(Y_1 - Y_0|X = x) \geq 0\}$:

- suppose individuals who benefit from treatment vote for it
- with $\tau = 0.5$, δ^* is a policy that obeys *majority vote*:

$$\begin{aligned} & Q_{0.5}(Y_1 - Y_0|X) \geq 0 \\ \Leftrightarrow \quad & F_{Y_1 - Y_0|X}(0) \leq 1/2 \\ \Leftrightarrow \quad & P[Y_1 \geq Y_0|X] \geq 1/2 \\ \Leftrightarrow \quad & P[Y_1 \geq Y_0|X] \geq P[Y_1 < Y_0|X] \end{aligned}$$

- consistent with a PM who has political incentive and whose decision is influenced by vote shares

This Paper: Quantile of Treatment Effects as Welfare Gain

- can be generalized by considering $Q_{0.5-\alpha/2}(Y_1 - Y_0|X) \geq 0$, which is equivalent to

$$P[Y_1 \geq Y_0|X] \geq P[Y_1 < Y_0|X] + \alpha$$

where $\alpha \geq 0$ is vote share margin

Optimal Policies Robust to Model Ambiguity

QoTE is generally not point-identified even under unconfoundedness

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We propose an optimal policy robust to model ambiguity:

$$\delta^*_{mmw} \in \arg\max_{\delta \in \mathcal{D}} \min_{F_{Y_1,Y_0|X} \in \mathcal{F}} E[\delta(X) Q_\tau(Y_1 - Y_0|X)]$$

- $\mathcal{F} \equiv \mathcal{F}(P)$ is the identified set of $F_{Y_1,Y_0|X}$ given data P

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Alternatively,

$$\delta^*_{mmr} \in \arg\min_{\delta \in \mathcal{D}} \max_{F_{Y_1,Y_0|X} \in \mathcal{F}} E[\{\delta^\dagger(X) - \delta(X)\} Q_\tau(Y_1 - Y_0|X)]$$

- $\delta^\dagger = 1\{Q_\tau(Y_1 - Y_0|\cdot) \geq 0\} \in \arg\max_\delta E[\delta(X) Q_\tau(Y_1 - Y_0|X)]$ is the first-best policy

Optimal Policies Robust to Model Ambiguity

Define the identified interval for $Q_\tau(Y_1 - Y_0|X = x)$:

$$[Q_\tau^L(x), Q_\tau^U(x)] = \{Q_\tau(Y_1 - Y_0|X = x) : F_{Y_1,Y_0|X} \in \mathcal{F}\}$$

Assumption REC: The identified set $\mathcal{Q}(P)$ of $Q_\tau(Y_1 - Y_0|X)$ is rectangular, i.e.,

$$\mathcal{Q}(P) \equiv \{Q_\tau(Y_1 - Y_0|X = \cdot) : Q_\tau(Y_1 - Y_0|X = x) \in [Q_\tau^L(x), Q_\tau^U(x)]\}$$

- allows to interchange the max/min over $\mathcal{F}$ with the expectation over X (Kasy 16, D'Adamo 23)

▸ REC

Optimal Policies Robust to Model Ambiguity

Under REC, we can show

$$\delta^*_{mmw} \in \arg\max_{\delta \in \mathcal{D}} E[\delta(X) Q^L_\tau(X)]$$

and

$$\delta^*_{mmr} \in \arg\max_{\delta \in \mathcal{D}} E\left[\delta(X)\Lambda(X)\right]$$

where

$$\begin{aligned}\Lambda(x) &= Q^U_\tau(x) \cdot 1\{Q^U_\tau(x) \geq 0\} + Q^L_\tau(x) \cdot 1\{Q^L_\tau(x) \leq 0\} \\ &= Q^U_\tau(x) \cdot 1\{Q^L_\tau(x) \geq 0\} + Q^L_\tau(x) \cdot 1\{Q^U_\tau(x) \leq 0\} \\ &\quad + \left(|Q^U_\tau(x)| - |Q^L_\tau(x)|\right) \cdot 1\{Q^L_\tau(x) < 0 < Q^U_\tau(x)\}\end{aligned}$$

Related Literature

Treatment choice and policy learning:

- Manski 04, 09, Hirano and Porter 09, Stoye 12, Kitagawa & Tetenov 18, Athey & Wager 21, Mbakop & Tabord-Meehan 21, Sakaguchi 21, Kitagawa, Sakaguchi, Tetenov 21, Ida, Ishihara, Ito, Kido, Kitagawa, Sakaguchi, Sasaki 22, among others
- Murphy, van der Laan & Robins 01, Murphy 03, Robins 04, Zhao, Zeng, Rush & Kosorok 12, Cui & Tchetgen Tchetgen 21, among others
- Wang et al. 18, Leqi & Kennedy 21, Kitagawa & Tetenov 21
- Kitagawa, Lee & Qiu 23

Treatment choice under ambiguity:

- Stoye 09, Kasy 16, Kallas & Zhou 21, Pu & Zhang 21, Cui 21, Yata 21, Han 23, D'Adamo 23

Possible Identifying Assumptions

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We provide a range of identifying assumptions that the researcher may want to impose

- to shrink $\mathcal{F}$ and thus $[Q_\tau^L(x), Q_\tau^U(x)]$,
- sometimes to a singleton

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First, to identify the marginal distribution of Y_d:

Assumption CI (Conditional Independence): $Y_d \perp D|X$ for $d \in \{0, 1\}$.

- Alternatively, panel quantile regression models can be used to identify $Q_\tau(Y_d|X)$ (Chernozhukov, Fernandez-Val, Hahn & Newey 13)

Possible Identifying Assumptions

No-assumption bounds on $Q_\tau(x)$ (besides CI):

- Makarov 81, Williamson & Downs 90
- may be uninformative

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Assumption SI (Stochastic Increasing): For given $x \in \mathcal{X}$, $P[Y_1 \leq y_1 | Y_0 = \cdot, X = x]$ and $P[Y_0 \leq y_0 | Y_1 = \cdot, X = x]$ are nonincreasing.

- SI + CI produce informative bounds
- Frandsen & Lefgren 21

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Assumption SD (Stochastic Dominance): For given $x \in \mathcal{X}$, (i) $P[Y_d \leq y | D = 1, X = x] \leq P[Y_d \leq y | D = 0, X = x]$; or (ii) $P[Y_1 \leq y | D = d, X = x] \leq P[Y_0 \leq y | D = d, X = x]$.

- SD(i) or SD(ii), without CI or with instruments
- Blundell, Gosling, Ichimura & Meghir 07, Lee 23

Possible Identifying Assumptions

Here are assumptions for point identification of $Q_\tau(x)$

Assumption CI2 (Joint Conditional Independence):
$(Y_1, Y_0) \perp D|X$.

Assumption DC (Deconvolution): $Y_1 - Y_0 \perp Y_0|X$.

- CI2 + DC point-identify $Q_\tau(x)$ (Heckman & Smith 95)

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Assumption RY (Roy Model): (i) $D = 1\{Y_1 \geq Y_0\}$; (ii) large support of elements of X; (iii) additive errors in Y_d-equations.

Assumption RY2 (Extended Roy Model):
(i) $D = 1\{Y_1 \geq h(Y_0, X, Z)\}$; (ii) strict monotonicity of h;
(iii) $(Y_0, Y_1) \perp Z|X$.

- RY or RY2 point-identifies $Q_\tau(x)$ (Heckman & Smith 95, Lee & Park 22)

Possible Identifying Assumptions

Assumption RI (Rank Invariance): (i) $Y_d = m_d(X, U_d)$; (ii) $m_d(x, \cdot)$ is strictly increasing; (iii) $U_1|_{X=x} = U_0|_{X=x}$.

- Heckman, Smith & Clements 97, Chernozhukov & Hansen 05
- generalized version in Heckman, Smith & Clements 97

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Assumption RY (Mutual Independence): $Y_1 \perp Y_0 | X, C$ for some variable C.

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- relates to factor models (Abbring & Heckman 07)

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Assumption SYM (Symmetric Distribution): The distribution of $Y_1 - Y_0 | X$ is symmetric.

- Then $Q_{0.5}(Y_1 - Y_0 | X) = E[Y_1 - Y_0 | X]$, which is point-identified under CI

Calculating Bounds on $Q_\tau(x)$

Let $C(u,v|X)$ be the copula for $(U,V) \equiv (F_{Y_1}(Y_1), F_{Y_0}(Y_0))$ conditional on X

Then, by Sklar's Theorem,

$$\begin{aligned} P[Y_1 - Y_0 \le t|X] &= P[F^{-1}_{Y_1|X}(U) - F^{-1}_{Y_0|X}(V) \le t|X] \\ &= \int 1\{F^{-1}_{Y_1|X}(u) - F^{-1}_{Y_0|X}(v) \le t\} dC(u,v|X) \end{aligned}$$

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Therefore, with $\Delta \equiv Y_1 - Y_0$,

$$F^L_{\Delta|X}(t) = \inf_{C(\cdot,\cdot|X)\in\mathcal{C}} \int 1\{F^{-1}_{Y_1|X}(u) - F^{-1}_{Y_0|X}(v) \leq t\} dC(u, v|X)$$

$$F^U_{\Delta|X}(t) = \sup_{C(\cdot,\cdot|X)\in\mathcal{C}} \int 1\{F^{-1}_{Y_1|X}(u) - F^{-1}_{Y_0|X}(v) \leq t\} dC(u, v|X)$$

where $\mathcal{C}$ is the class of copulas restricted by identifying assumptions

Calculating Bounds on $Q_\tau(x)$

For τ-quantile $Q_\tau(x)$ for $\Delta|X = x$, we can obtain its lower and upper bounds as

$$Q^L_\tau(X) = F^{U,-1}_{\Delta|X}(\tau)$$
$$Q^U_\tau(X) = F^{L,-1}_{\Delta|X}(\tau)$$

In practice, we need to approximate $C(u, v|x)$ to transform above optimization into linear programs

- two approaches

Calculating Bounds on $Q_\tau(x)$: Approach I

For Makarov bounds, consider (suppressing X)

$$F^L_\Delta(t) = \min_{c(\cdot,\cdot)} \sum_{j=1}^{k} \sum_{i=1}^{k} 1\{F^{-1}_{Y_1}(r(i)) - F^{-1}_{Y_0}(r(j)) \leq t\} c(i,j)$$

$$F^U_\Delta(t) = \max_{c(\cdot,\cdot)} \sum_{j=1}^{k} \sum_{i=1}^{k} 1\{F^{-1}_{Y_1}(r(i)) - F^{-1}_{Y_0}(r(j)) \leq t\} c(i,j)$$

where

$$r(i) = \frac{2i-1}{2k}$$

and

$$\sum_{s=1}^{k} c(s,j) = 1/k, \text{for } j = 1 \ldots k$$

$$\sum_{s=1}^{k} c(i,s) = 1/k, \text{for } i = 1 \ldots k$$

Calculating Bounds on $Q_\tau(x)$: Approach I

Additionally, e.g., Assumption SI imposes

$$\left\{ \left\{ \sum_{s=1}^{i} c(s,j) \geq \sum_{s=1}^{i} c(s,j+1) \right\}_{j=1}^{k-1} \right\}_{i=1}^{k}$$

$$\left\{ \left\{ \sum_{s=1}^{i} c(i,s) \geq \sum_{s=1}^{j} c(i+1,s) \right\}_{i=j}^{k-1} \right\}_{j=1}^{k}$$

Calculating Bounds on $Q_\tau(x)$: Approach II

Alternatively, we can approximate $C(u, v|x)$ using Bernstein copula $C_B(u, v|x)$ (Sancetta & Satchell 04)

Finally, $F_{Y_1|X}(y)$ and $F_{Y_0|X}(y)$ can be estimated using standard nonparametric or parametric methods

Theoretical Properties of Estimated Policy

Recall $Q_\tau(X) \equiv Q_\tau(Y_1 - Y_0|X)$ and our objective function is

$$V(\delta) \equiv E[\delta(X)Q_\tau(X)]$$

The regret of this "classification" is

$$R(\delta) \equiv V(\delta^\dagger) - V(\delta) = E[|Q_\tau(X)|1\{\delta(X) \neq sign(Q_\tau(X))\}]$$

- $sign(q) = 1$ when $q \geq 0$ and $sign(q) = 0$ when $q < 0$

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$R(\delta)$ is not identified, thus we define maximum regret as

$$\bar{R}(\delta) \equiv \sup_{Q_\tau(\cdot) \in [Q_\tau^L(\cdot), Q_\tau^U(\cdot)]} E[|Q_\tau(X)| 1\{\delta(X) \neq sign(Q_\tau(X))\}]$$

Focus on the case where $\mathcal{D}$ is unrestricted

Theoretical Properties of Estimated Policy

Assumption EST: $F^{-1}_{\Delta|X}(\tau)$ is bounded a.s. and

$$\hat{Q}^L_\tau(X) - Q^L_\tau(X) = o_p(1),$$
$$\hat{Q}^U_\tau(X) - Q^U_\tau(X) = o_p(1).$$

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Theorem 1 (Regret Bounds)

Suppose EST holds. Then,

$$R(\hat{\delta}^{stoch}) \leq \bar{R}(\hat{\delta}^{stoch}) \leq E\left[\frac{Q^L_\tau(X)Q^U_\tau(X)}{Q^L_\tau(X)-Q^U_\tau(X)}1\{Q^L_\tau(X)<0<Q^U_\tau(X)\}\right]+o_p(1),$$

where the ratio $= 0$ when its denominator $= 0$, and

$$R(\hat{\delta}^{determ}) \leq \bar{R}(\hat{\delta}^{determ}) \leq E\left[\min(\max(Q^U_\tau(X),0),\max(-Q^L_\tau(X),0))\right]+o_p(1).$$

Theoretical Properties of Estimated Policy

Corollary 1 (Expected Regret Bounds)

Suppose EST holds. Then,

$$E_{P^n}\left[R(\hat{\delta}^{stoch})\right] \leq E\left[\frac{Q^L_\tau(X)Q^U_\tau(X)}{Q^L_\tau(X)-Q^U_\tau(X)}1\{Q^L_\tau(X)<0<Q^U_\tau(X)\}\right]+o(1),$$

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Theoretical Properties of Estimated Policy

Corollary 1 (Expected Regret Bounds)

Suppose EST holds. Then,

$$E_{P^n}\left[R(\hat{\delta}^{stoch})\right] \leq E\left[\frac{Q_\tau^L(X)Q_\tau^U(X)}{Q_\tau^L(X)-Q_\tau^U(X)}1\{Q_\tau^L(X)<0<Q_\tau^U(X)\}\right]+o(1),$$

where the ratio $=0$ when its denominator $=0$, and

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Leading term in each bound reduces to zero when either...

- bounds on $Q_\tau(X)$ excludes zero a.s.
- or $Q_\tau(X)$ is point-identified

Classification Method with Constrained Policy Class

Sometimes PM may be interested in parsimonious decision rules

- e.g., threshold policies with linear index

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$$\delta^*_{mmw} \in \arg\max_{\delta \in \mathcal{D}} E[\delta(X) Q^L_\tau(X)]$$

$$\delta^*_{mmr} \in \arg\max_{\delta \in \mathcal{D}} E[\delta(X)[Q^U_\tau(X) \cdot 1\{Q^U_\tau(X) \geq 0\} + Q^L_\tau(X) \cdot 1\{Q^L_\tau(X) \leq 0\}]]$$

We can consider convex relaxation by using hinge loss function $\phi(t) = \max(1 - t, 0)$ and adding regularization

- e.g., the outcome weighted learning framework (Zhao et al. 12)
- consistency with hinge loss is proved even when the classifier's prediction set is restricted (Kitagawa et al., 2021)

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- e.g., the outcome weighted learning framework (Zhao et al. 12)
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Then bound on $\bar{R}$ and thus bound on R can be obtained

Numerical Illustrations

Q: How policies differ across PM's criteria esp. when the QoTE is partially identified?

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Data-generating process:

- Draw (Y_1, Y_0) (or $(\log Y_1, \log Y_0)$) from $N(\mu, \Sigma)$ with $\mu = (\mu_1, \mu_0)'$ and

$$\Sigma = \begin{bmatrix} \sigma_1 & \rho_{10}\sqrt{\sigma_0\sigma_1} \\ \rho_{10}\sqrt{\sigma_0\sigma_1} & \sigma_0 \end{bmatrix}$$

- Draw D from $Bernoulli(0.5)$
- Generate $Y = DY_1 + (1 - D)Y_0$

Note: $Y_1|Y_0 \sim N(\mu_1 + \rho_{10}\sigma_1 Z_0, (1 - \rho^2\sigma_1))$ where $Z_0 = \frac{Y_0 - \mu_0}{\sigma_0}$

- thus SI holds when $\rho_{10} \geq 0$ (similarly for log-normal case)

Numerical Illustrations

When $\mathcal{D}$ is unrestricted, the true optimal policies are:

- $\delta^* = 1\{Q_\tau(Y_1 - Y_0) > 0\}$
  - e.g., $Q_\tau(Y_1 - Y_0) = \mu_1 - \mu_0 + \Phi^{-1}(\tau)\sqrt{\sigma_1^2 + \sigma_0^2 - 2\rho_{10}\sigma_1\sigma_0}$
- $\delta^*_{QTE} = 1\{Q_\tau(Y_1) - Q_\tau(Y_0) > 0\}$
  - e.g., $Q_\tau(Y_1) - Q_\tau(Y_0) = \mu_1 - \mu_0 + \Phi^{-1}(\tau)(\sigma_1 - \sigma_0)$
- $\delta^*_{ATE} = 1\{E[Y_1 - Y_0] > 0\}$
  - e.g., $E[Y_1 - Y_0] = \mu_1 - \mu_0$
- they are first best for both deterministic and stochastic policies
- under normality and SI, if $0 < \tau < 0.5$, we have...
  - $Q_\tau(Y_1 - Y_0) < Q_\tau(Y_1) - Q_\tau(Y_0)$ and
  - $Q_\tau(Y_1 - Y_0) < E(Y_1) - E(Y_0)$

For brevity, let $Q_\tau \equiv Q_\tau(Y_1 - Y_0)$

Numerical Illustrations

Unlike δ^*_{QTE} and δ^*_{ATE}, recall obtaining $\delta^* \equiv \delta^*_{QoTE}$ involves model uncertainty:

- for deterministic policy:

$$\delta^*_{mmr} = \begin{cases} 1\{|Q^U_\tau| \geq |Q^L_\tau|\} & \text{if } Q^L_\tau < 0 < Q^U_\tau \\ 1 & \text{if } Q^L_\tau \geq 0 \\ 0 & \text{if } Q^U_\tau \leq 0 \end{cases}$$

- for stochastic policy:

$$\delta^*_{mmr} \sim \begin{cases} Bernoulli\left(\frac{Q_U}{Q_U - Q_L}\right) & \text{if } Q^L_\tau < 0 < Q^U_\tau \\ 1 & \text{if } Q^L_\tau \geq 0 \\ 0 & \text{if } Q^U_\tau \leq 0 \end{cases}$$

In simulation, Q^L_τ and Q^U_τ are calculated under...

- no assumption (i.e., Makarov bounds) or Assumption SI

Numerical Illustrations

For δ^*_{mmr}, δ^*_{QTE} and δ^*_{ATE}, we estimate $\hat{\delta}^*$, $\hat{\delta}^*_{QTE}$ and $\hat{\delta}^*_{ATE}$

- by estimating Q^U_τ, Q^L_τ, $Q_\tau(Y_d)$, and $E[Y_d]$ $(d = 0, 1)$ using the generated experimental data

For $TE \in \{QoTE, QTE, ATE\}$, (recalling $\delta^* \equiv \delta^*_{QoTE}$)

- misclassification error: $E_{P^n}[1\{\hat{\delta}_{TE} \neq \delta^*_{TE}\}]$
- regret: $E_{P^n}[|TE| \cdot 1\{\hat{\delta}_{TE} \neq \delta^*_{TE}\}]$

We focus on $\tau = 0.25$

Numerical Results: Correct Classification Rate ($n = 50$)

	$\hat{\delta}^{SI,stoch}$	$\hat{\delta}^{stoch}$	$\hat{\delta}^{SI,determ}$	$\hat{\delta}^{determ}$	$\hat{\delta}_{QTE}$	$\hat{\delta}_{ATE}$
	Subgroup 1 (0 is not contained)					
δ^*	100%	100%	100%	100%	33.5%	93%
δ^*_{QTE}	0%	0%	0%	0%	66.5%	7%
δ^*_{ATE}	100%	100%	100%	100%	33.5%	93%
	Subgroup 2 (0 is not contained)					
δ^*	92%	90%	90.5%	94%	1%	17%
δ^*_{QTE}	8%	10%	9.5%	6%	99%	83%
δ^*_{ATE}	8%	10%	9.5%	6%	99%	83%

Numerical Results: Correct Classification Rate ($n = 50$)

	$\hat{\delta}^{SI,stoch}$	$\hat{\delta}^{stoch}$	$\hat{\delta}^{SI,determ}$	$\hat{\delta}^{determ}$	$\hat{\delta}_{QTE}$	$\hat{\delta}_{ATE}$
	Subgroup 3 (0 is contained)					
δ^*	79%	51%	84%	62%	99.5%	100%
δ^*_{QTE}	79%	51%	84%	62%	99.5%	100%
δ^*_{ATE}	79%	51%	84%	62%	99.5%	100%
	Subgroup 4 (0 is contained)					
δ^*	26%	49.5%	14.5%	44%	1.5%	0.5%
δ^*_{QTE}	74%	50.5%	85.5%	56%	98.5%	99.5%
δ^*_{ATE}	74%	50.5%	85.5%	56%	98.5%	99.5%
	Subgroup 5 (0 is contained, SI false)					
δ^*	82%	81%	83%	93.5%	66.5%	4%
δ^*_{QTE}	82%	81%	83%	93.5%	66.5%	4%
δ^*_{ATE}	18%	19%	17%	6.5%	33.5%	96%
	Subgroup 6 (0 is contained, SI false)					
δ^*	43%	61.5%	40%	61%	24%	0%
δ^*_{QTE}	57%	38.5%	60%	39%	76%	100%
δ^*_{ATE}	57%	38.5%	60%	39%	76%	100%

Numerical Results: Correct Classification Rate ($n = 50$)

	$\hat{\delta}^{SI,stoch}$	$\hat{\delta}^{stoch}$	$\hat{\delta}^{SI,determ}$	$\hat{\delta}^{determ}$	$\hat{\delta}_{QTE}$	$\hat{\delta}_{ATE}$
	Subgroup 8 (log normal, SI excludes 0)					
δ^*	95.5%	68%	95.5%	73%	100%	76.5%
δ^*_{QTE}	95.5%	68%	95.5%	73%	100%	76.5%
δ^*_{ATE}	4.5%	32%	4.5%	28%	0%	23.5%

Numerical Results: Correct Classification Rate ($n = 1000$)

	$\hat{\delta}^{SI,stoch}$	$\hat{\delta}^{stoch}$	$\hat{\delta}^{SI,determ}$	$\hat{\delta}^{determ}$	$\hat{\delta}_{QTE}$	$\hat{\delta}_{ATE}$
	Subgroup 1 (0 is not contained)					
δ^*	100%	100%	100%	100%	0%	100%
δ^*_{QTE}	0%	0%	0%	0%	100%	0%
δ^*_{ATE}	100%	100%	100%	100%	0%	100%
	Subgroup 2 (0 is not contained)					
δ^*	100%	99%	100%	100%	0%	0%
δ^*_{QTE}	0%	1%	0%	0%	100%	100%
δ^*_{ATE}	0%	1%	0%	0%	100%	100%

Numerical Results: Correct Classification Rate ($n = 1000$)

	$\hat{\delta}^{SI,stoch}$	$\hat{\delta}^{stoch}$	$\hat{\delta}^{SI,determ}$	$\hat{\delta}^{determ}$	$\hat{\delta}_{QTE}$	$\hat{\delta}_{ATE}$
	Subgroup 3 (0 is contained)					
δ^*	89%	59%	100%	95%	100%	100%
δ^*_{QTE}	89%	59%	100%	95%	100%	100%
δ^*_{ATE}	89%	59%	100%	95%	100%	100%
	Subgroup 4 (0 is contained)					
δ^*	21%	43%	0%	20%	0%	0%
δ^*_{QTE}	79%	57%	100%	80%	100%	100%
δ^*_{ATE}	79%	57%	100%	80%	100%	100%
	Subgroup 5 (0 is contained, SI false)					
δ^*	92%	74%	100%	100%	100%	0%
δ^*_{QTE}	92%	74%	100%	100%	100%	0%
δ^*_{ATE}	8%	26%	0%	0%	0%	100%
	Subgroup 6 (0 is contained, SI false)					
δ^*	34%	48%	6.5%	86%	0%	0%
δ^*_{QTE}	66%	52%	93.5%	14%	100%	100%
δ^*_{ATE}	66%	52%	93.5%	14%	100%	100%

Numerical Results: Correct Classification Rate ($n = 1000$)

	$\hat{\delta}^{SI,stoch}$	$\hat{\delta}^{stoch}$	$\hat{\delta}^{SI,determ}$	$\hat{\delta}^{determ}$	$\hat{\delta}_{QTE}$	$\hat{\delta}_{ATE}$
	Subgroup 8 (log normal, SI excludes 0)					
δ^*	100%	60%	100%	97%	100%	70.5%
δ^*_{QTE}	100%	60%	100%	97%	100%	70.5%
δ^*_{ATE}	0%	40%	0%	3%	0%	29.5%

Application I: Allocations of Right Heart Catheterization

We consider the right heart catheterization (RHC) (e.g., Hirano & Imbens 02)

- D: RHC (1 if received and 0 otherwise), a diagnostic procedure for critically ill patients
- Y: number of days from admission to death within 30 days

Studies like Connors et al. 21 found that patient survival is lower with RHC than without

- therefore, relevant policy question is to find patients for whom allocating (or avoiding!) RHC is life-saving

In the dataset, 5735 patients are divided into a treatment group (2184 patients) and a control group (3551 patients)

- X: age, sex, coma, DNR status, est'ed survival rate, ICU mortality score

Application I: Allocations of Right Heart Catheterization

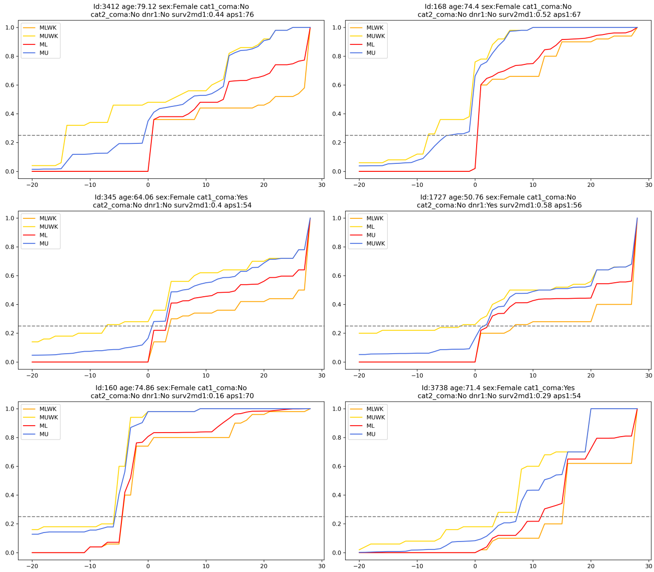


Figure: Bounds on the QoTE of Six Representative Patients ($\tau = 0.25$)

Application II: Allocations of Job Training

We consider the National Job Training Partnership Act (JTPA) Study (Bloom et al. 97, Abadie et al. 02, Kitagawa & Tetenov 18)

We use a subset that includes 9,223 adults; 6,133 received job training, while 3,090 did not

- D: job training
- Y: 30-month earnings after job training
- X: sex, years of education, high school diploma, previous earnings

Q: how do prudent/negligent policies look like, relative to e.g. utilitarian policy?

Application II: Allocations of Job Training

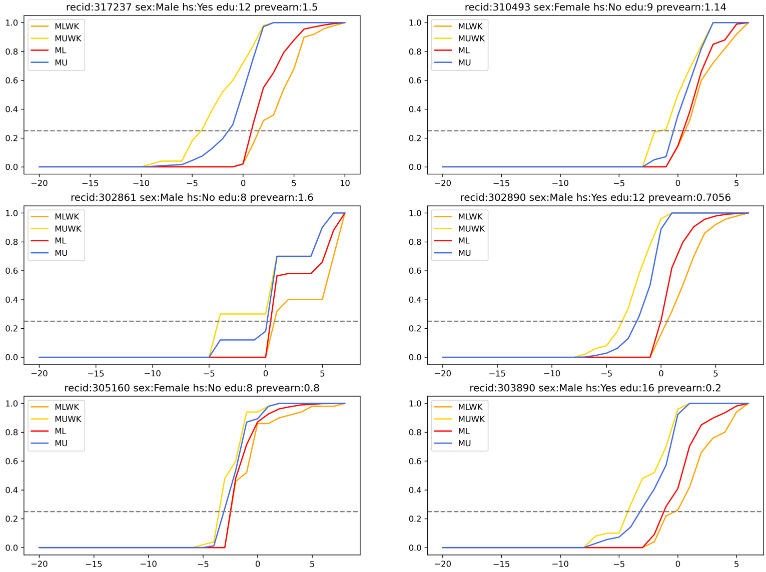
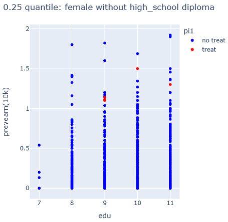
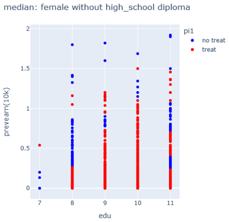


Figure: Bounds on the QoTE of Six Representative Workers ($\tau = 0.25$)

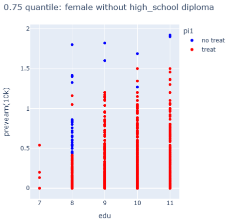
Application II: Allocations of Job Training



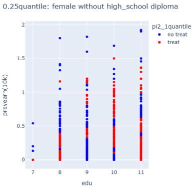
(a) $\tau = 0.25$



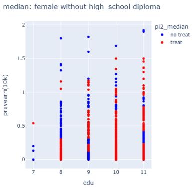
(b) $\tau = 0.5$



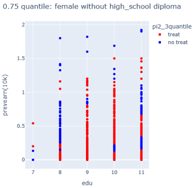
(c) $\tau = 0.75$



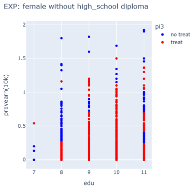
(d) 0.25-QTE



(e) 0.5-QTE



(f) 0.75-QTE



(g) ATE

Figure: Decisions for Female Workers Without High School Diploma

Concluding Remarks

The paper...

- proposes optimal allocation decisions
- under welfare gain defined by quantile of treatment effects
- that can also be motivated by vote shares;
- considers ambiguity-robust decisions;
- provides theoretical guarantee by calculating regret bounds;
- proposes a range of identifying assumptions

Extensions:

- interquartile range as equity target for an egalitarian PM
  - Kitagawa & Tetenov 21
- $E[Y_1 - Y_0 | Y_0 < c]$ as alternative prioritarian objective

Thank You! ☺

Rectangular Identified Set

Consider $X \in \{0, 1\}$

REC imposes that

$$\{(Q_\tau(0), Q_\tau(1)) : Q_\tau(x) \in [Q^L_\tau(x), Q^U_\tau(x)], x \in \{0, 1\}\}$$

is rectangular

Then

$$\begin{aligned}
\min_{F_{Y_1,Y_0|X}} E[\delta(X) Q_\tau(X)] &= \min\,[p_1 \delta(1) Q_\tau(1) + p_0 \delta(0) Q_\tau(0)] \\
&= p_1 \delta(1) \min Q_\tau(1) + p_0 \delta(0) \min Q_\tau(0) \\
&= p_1 \delta(1) Q^L_\tau(1) + p_0 \delta(0) Q^L_\tau(0) \\
&= E[\delta(X) \min Q_\tau(X)]
\end{aligned}$$

Return