Supplement to "Multiple Treatments with Strategic Substitutes"

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Abstract

In this supplemental appendix, we present all the proofs of theorems and lemmas in the main text and a further description of the data we use in the application.

1 Proofs

In terms of notation, when no confusion arises, we sometimes change the order of entry and write $\boldsymbol{v} = (v_s, \boldsymbol{v}_{-s})$ for convenience. For a multivariate function $f(\boldsymbol{v})$, the integral $\int_A f(\boldsymbol{v})d\boldsymbol{v}$ is understood as a multi-dimensional integral over a set A contained in the space of $\boldsymbol{v}$. Vectors in this paper are row vectors. Also, we write $Y_{\boldsymbol{d}} \equiv Y(\boldsymbol{d})$ for simplicity in this section.

1.1 Proof of Theorem 3.1

We prove the theorem by showing the following lemma:

Lemma 1.1. *Under Assumptions SS, for $j = 0, ..., S-1$, $\boldsymbol{R}^{\leq j}(\boldsymbol{z})$ is expressed as a union across $\sigma(\cdot) \in \Sigma$ of Cartesian products, each of which is a product of intervals that are either $(0, 1]$ or $\left(\nu^{\sigma(s)}(d^j_{-\sigma(s)}, z_{\sigma(s)}), 1\right]$ for some $s = 1, ...S$.*

Given this lemma, (3.5) holds by Assumption M1, because for given $s, \left(\nu^s(d^j_{-s}, z_s), 1\right] \subseteq \left(\nu^s(d^j_{-s}, z'_s), 1\right]$ for any d^j_{-s} where the direction of inclusion is given by (3.4). Now we prove Lemma 1.1.

Consider $d_s^j = 1$ for an s-th element d_s^j in $\boldsymbol{d}^j$ $(j \geq 1)$. Then there exists $\boldsymbol{d}^{j-1}$ such that $d_s^{j-1} = 0$. Suppose not. Then $d_s^{j-1} = 1$ $\forall \boldsymbol{d}^{j-1}$, and thus we can construct $\boldsymbol{d}^{j-1}$ that is equal to $\boldsymbol{d}^j$, which is contradiction. Therefore, in calculating $\boldsymbol{R}_j(\boldsymbol{z}) \cup \boldsymbol{R}_{j-1}(\boldsymbol{z})$, according to (D.2), what is involved is the union of intervals associated with $d_s^j = 1$ and $d_s^{j-1} = 0$, while sharing the same opponent d_{-s}^{j-1}: $\left(0, \nu^s(d_{-s}^{j-1}, z_s)\right] \cup \left(\nu^s(d_{-s}^{j-1}, z_s), 1\right] = (0, 1]$. This implies that $\boldsymbol{R}_j(\boldsymbol{z}) \cup \boldsymbol{R}_{j-1}(\boldsymbol{z})$ is not a function of $\boldsymbol{z}$ through $\nu^s(d_{-s}^{j-1}, z_s)$ for any, and nor is $\boldsymbol{R}^{\leq j}(\boldsymbol{z}) \equiv \bigcup_{k=0}^{j} \boldsymbol{R}_k(\boldsymbol{z})$. On the other hand, when $d_s^j = 0$ for given s, the associated interval is $\left(\nu^s(d_{-s}^j, z_s), 1\right]$ as shown in (D.2). Therefore, $\boldsymbol{R}^{\leq j}(\boldsymbol{z}) \equiv \bigcup_{k=0}^{j} \boldsymbol{R}_k(\boldsymbol{z})$ is a function of $\boldsymbol{z}$ only through $\nu^s(d_{-s}^j, z_s)$ for some s. This proves Lemma 1.1.

1.2 Proof of Lemma 3.1

The following proposition is useful later:

Proposition 1.1. *Let R and Q be sets defined by Cartesian products: $R = \prod_{s=1}^{S} r_s$ and $Q = \prod_{s=1}^{S} q_s$ where r_s and q_s are intervals in $\mathbb{R}$. Then $R \cap Q = \prod_{s=1}^{S} r_s \cap q_s$.*

The proof of this proposition follows directly from the definition of R and Q.

The first part proves that Assumption EQ is equivalent to $R_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}') = \emptyset$ for all $\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j$ and j. For any $\boldsymbol{d}^j$ and $\tilde{\boldsymbol{d}}^j$ $(\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j)$, the expression of $R_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}')$ can be inferred as follows. Under Assumption M1, we can simplify the notation of the payoff function as $\nu_j^s(z_s) \equiv \nu^s(\boldsymbol{d}_{-s}^j, z_s)$ when we compare it for different values of z_s. First, there exists s^* such that $d_{s^*}^j = 1$ and $\tilde{d}_{s^*}^j = 0$ (without loss of generality), otherwise it contradicts $\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j$. That is, $U_{s^*} \in \left(0, \nu_{j-1}^{s^*}(z_{s^*})\right]$ in $R_{\boldsymbol{d}^j}(\boldsymbol{z})$ and $U_{s^*} \in \left(\nu_j^{s^*}(z'_{s^*}), 1\right]$ in $R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}')$. For other $s \neq s^*$, the pair is realized to be one of the four types: (i) $d_s^j = 1$ and $\tilde{d}_s^j = 0$; (ii) $d_s^j = 0$ and $\tilde{d}_s^j = 1$; (iii) $d_s^j = 1$ and $\tilde{d}_s^j = 1$; (iv) $d_s^j = 0$ and $\tilde{d}_s^j = 0$. Then the corresponding pair of intervals for $R_{\boldsymbol{d}^j}(\boldsymbol{z})$ and $R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}')$, respectively, falls into one of the four types: (i) $\left(0, \nu_{j-1}^s(z_s)\right]$ and $\left(\nu_j^s(z'_s), 1\right]$; (ii) $\left(\nu_j^s(z_s), 1\right]$ and $\left(0, \nu_{j-1}^s(z'_s)\right]$; (iii) $\left(0, \nu_{j-1}^s(z_s)\right]$ and $\left(0, \nu_{j-1}^s(z'_s)\right]$; (iv) $\left(\nu_j^s(z_s), 1\right]$ and $\left(\nu_j^s(z'_s), 1\right]$. Then by Proposition 1.1, $R_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}')$ is a product of the intersections of the interval pairs. But the intersection resulting from $\left(0, \nu_{j-1}^{s^*}(z_{s^*})\right]$ and $\left(\nu_j^{s^*}(z'_{s^*}), 1\right]$ is empty if and only if $\nu_{j-1}^{s^*}(z_{s^*}) \leq \nu_j^{s^*}(z'_{s^*})$. Therefore, $R_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}') = \emptyset$ for all $\boldsymbol{d}^j$ and $\tilde{\boldsymbol{d}}^j$ $(\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j)$ if and only if $(\boldsymbol{z}, \boldsymbol{z}')$ are such that $\nu_{j-1}^s(z_s) \leq \nu_j^s(z'_s)$ for all s. Additionally, note that $R_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}') = \emptyset$ implies

$$R^*_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R^*_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}') = R^*_{\boldsymbol{d}^j}(\boldsymbol{z}') \cap R^*_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}) = \emptyset \tag{1.1}$$

for $\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j$, where $R^*_{\boldsymbol{d}}(\boldsymbol{z})$ is the region that predicts equilibrium $\boldsymbol{d}$.[1] This last display is useful later in other proofs later.

Moreover, note that any region $R^M_j(\boldsymbol{z})$ of multiple equilibria for $\mathcal{D}_j$ given $\boldsymbol{z}$ is defined by the intersection of the following interval pairs (and no more): (i) $\left(0, \nu^s_{j-1}(z_s)\right]$ and $\left(\nu^s_j(z_s), 1\right]$; (ii) $\left(0, \nu^s_{j-1}(z_s)\right]$ and $\left(0, \nu^s_{j-1}(z_s)\right]$; (iii) $\left(\nu^s_j(z_s), 1\right]$ and $\left(\nu^s_j(z_s), 1\right]$. Therefore, by Assumption SS (i.e., $\nu^s_{j-1}(z_s) > \nu^s_j(z_s)$), such a region is defined by the following corresponding intersections: (i) $\left(\nu^s_j(z_s), \nu^s_{j-1}(z_s)\right]$; (ii) $\left(0, \nu^s_{j-1}(z_s)\right]$; (iii) $\left(\nu^s_j(z_s), 1\right]$. Therefore $R^M_j(\boldsymbol{z}) \cap R^M_j(\boldsymbol{z}') = \emptyset$ if and only if $R_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}') = \emptyset$ for $\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j$.

We now prove that, when (3.6) holds, it satisfies $R^M_j(\boldsymbol{z}) \cap R^M_j(\boldsymbol{z}') = \emptyset$ for all j. We first prove the claim for $S = 2$ and then generalize it. The probabilities in (3.6) equal

$$\Pr[\boldsymbol{D} = (1,1) | \boldsymbol{Z} = \boldsymbol{z}] = \Pr[\boldsymbol{U} \in R_{11}(\boldsymbol{z})],$$

$$\Pr[\boldsymbol{D} = (0,0) | \boldsymbol{Z} = \boldsymbol{z}'] = \Pr[\boldsymbol{U} \in R_{00}(\boldsymbol{z}')].$$

Under independent unobserved types, these probabilities are equivalent to the volume of $R_{11}(\boldsymbol{z})$ and $R_{00}(\boldsymbol{z}')$, respectively. We consider two isoquant curves that are subsets of the surface of circles in $\mathcal{U}$: a curve $C_{11}(\boldsymbol{z})$ that is strictly convex from its origin $(0,0)$ and delivers the same volume as $R_{11}(\boldsymbol{z})$ and a curve $C_{00}(\boldsymbol{z}')$ that is strictly convex from its origin $(1,1)$ for $R_{00}(\boldsymbol{z}')$. Note that any region of multiple equilibria lies between the curve and its *opposite* origin. That is, $R^M_1(\boldsymbol{z})$ lies between $C_{11}(\boldsymbol{z})$ and $(1,1)$, and $R^M_1(\boldsymbol{z}')$ lies between $C_{00}(\boldsymbol{z}')$ and $(0,0)$. Therefore, if $C_{11}(\boldsymbol{z}) \cap C_{00}(\boldsymbol{z}') = \emptyset$ then $R^M_1(\boldsymbol{z}) \cap R^M_1(\boldsymbol{z}') = \emptyset$, because the curves are strictly convex.

The remaining argument is to prove that $C_{11}(\boldsymbol{z}) \cap C_{00}(\boldsymbol{z}') = \emptyset$. In order for this to be true, the sum of the radii of $C_{11}(\boldsymbol{z})$ and $C_{00}(\boldsymbol{z}')$ should not be great than $\sqrt{2}$, the length of the *space diagonal* of $\mathcal{U} = (0,1]^2$. But note that the radius can be identified from the data by considering an extreme scenario along each isoquant curve. First, consider the situation that player 1 is unprofitable to

[1] Note that $R^*_{\boldsymbol{d}}(\boldsymbol{z})$ is unknown to the econometrician even if all the players' payoffs had been known, since the equilibrium selection rule is unknown. This is in contrast to $R_{\boldsymbol{d}}(\boldsymbol{z})$ defined in Section D, which is purely determined by the payoffs $\nu^s_{\boldsymbol{d}_{-s}}(z_s)$,

enter irrespective of player 2's decisions with $\boldsymbol{z}$. Then $\mathcal{U} = \tilde{R}_{11}(\boldsymbol{z}) \cup \tilde{R}_{10}(\boldsymbol{z})$ and it is easy to see that $1 - \Pr[\boldsymbol{U} \in \tilde{R}_{11}(\boldsymbol{z})]$ the radius of $C_{11}(\boldsymbol{z})$. Second, consider a situation that player 1 is profitable to enter irrespective of player 2's decisions with $\boldsymbol{z}'$. Then $\mathcal{U} = \tilde{R}_{00}(\boldsymbol{z}') \cup \tilde{R}_{01}(\boldsymbol{z}')$ and $1 - \Pr[\boldsymbol{U} \in \tilde{R}_{00}(\boldsymbol{z}')]$ is the radius of $C_{00}(\boldsymbol{z}')$. Therefore, $C_{11}(\boldsymbol{z}) \cap C_{00}(\boldsymbol{z}') = \emptyset$ is implied by

$$\begin{aligned}\sqrt{2} &> (1 - \Pr[\boldsymbol{U} \in \tilde{R}_{11}(\boldsymbol{z})]) + (1 - \Pr[\boldsymbol{U} \in \tilde{R}_{00}(\boldsymbol{z}')]) \\ &= (1 - \Pr[\boldsymbol{U} \in R_{11}(\boldsymbol{z})]) + (1 - \Pr[\boldsymbol{U} \in R_{00}(\boldsymbol{z}')]),\end{aligned}$$

where the equality is by the definition of the isoquant curves.

To prove the general case for $S \geq 2$, we iteratively apply the result from the previous case of one less player, starting from $S = 2$. Suppose $S = 3$. Consider $R_{111}(\boldsymbol{z})$ and $R_{001}(\boldsymbol{z}')$. By definition, these regions are analogous to the regions in the $S = 2$ case above on the surface $\{(U_1, U_2, 0)\} \subset \mathcal{U} = (0, 1]^3$. Similarly, the following is the pairs of regions and corresponding surfaces that are analogous to $S = 2$: $R_{110}(\boldsymbol{z})$ and $R_{000}(\boldsymbol{z}')$ on $\{(U_1, U_2, 1)\}$, $R_{111}(\boldsymbol{z})$ and $R_{010}(\boldsymbol{z}')$ on $\{(U_1, 0, U_3)\}$, $R_{101}(\boldsymbol{z})$ and $R_{000}(\boldsymbol{z}')$ on $\{(U_1, 1, U_3)\}$, $R_{111}(\boldsymbol{z})$ and $R_{100}(\boldsymbol{z}')$ on $\{(0, U_2, U_3)\}$, $R_{011}(\boldsymbol{z})$ and $R_{000}(\boldsymbol{z}')$ on $\{(1, U_2, U_3)\}$. But note that any region of multiple equilibria can be partitioned and projected on the regions of multiple equilibria on these surface; see Figures 9 and 10. Therefore, $R_j^M(\boldsymbol{z}) \cap R_j^M(\boldsymbol{z}') = \emptyset$ for all j if

$$\begin{aligned}\sqrt{2} &> (1 - \Pr[\boldsymbol{U} \in R_{\boldsymbol{d}^j}(\boldsymbol{z})]) + (1 - \Pr[\boldsymbol{U} \in R_{\boldsymbol{d}^{j-2}}(\boldsymbol{z}')]) \\ &= (1 - \Pr[\boldsymbol{D} = \boldsymbol{d}^j | \boldsymbol{z}]) + (1 - \Pr[\boldsymbol{D} = \boldsymbol{d}^{j-2} | \boldsymbol{z}']) \end{aligned} \tag{1.2}$$

for all $\boldsymbol{d}^j$ and $\boldsymbol{d}^{j-2}$ and $j \in \{2, 3\}$. Next, for $S = 4$, focusing on the surfaces of the hypercube $\mathcal{U} = (0, 1]^4$, we can apply the result from $S = 3$, and so on. Therefore, in general, $R_j^M(\boldsymbol{z}) \cap R_j^M(\boldsymbol{z}') = \emptyset$ for all j if (1.2) for any $\boldsymbol{d}^j \in \mathcal{D}^j$, $\boldsymbol{d}^{j-2} \in \mathcal{D}^{j-2}$ and $2 \leq j \leq S$.

1.3 Proof of Result (2.13)

Introduce

$$h_{11}(\boldsymbol{z},\boldsymbol{z}') \equiv \Pr[Y=1,\boldsymbol{D}=(1,1)|\boldsymbol{Z}=\boldsymbol{z}] - \Pr[Y=1,\boldsymbol{D}=(1,1)|\boldsymbol{Z}=\boldsymbol{z}'],$$

$$h_{00}(\boldsymbol{z},\boldsymbol{z}') \equiv \Pr[Y=1,\boldsymbol{D}=(0,0)|\boldsymbol{Z}=\boldsymbol{z}] - \Pr[Y=1,\boldsymbol{D}=(0,0)|\boldsymbol{Z}=\boldsymbol{z}'],$$

$$h_{10}(\boldsymbol{z},\boldsymbol{z}') \equiv \Pr[Y=1,\boldsymbol{D}=(1,0)|\boldsymbol{Z}=\boldsymbol{z}] - \Pr[Y=1,\boldsymbol{D}=(1,0)|\boldsymbol{Z}=\boldsymbol{z}'],$$

$$h_{01}(\boldsymbol{z},\boldsymbol{z}') \equiv \Pr[Y=1,\boldsymbol{D}=(0,1)|\boldsymbol{Z}=\boldsymbol{z}] - \Pr[Y=1,\boldsymbol{D}=(0,1)|\boldsymbol{Z}=\boldsymbol{z}'].$$

Then h defined in (2.9) satisfies $h = h_{11} + h_{00} + h_{10} + h_{01}$. Let R^*_{10} and R^*_{01} be the regions that predict $\boldsymbol{D}=(1,0)$ and $\boldsymbol{D}=(0,1)$, respectively, which is unknown since the equilibrium selection mechanism is unknown. Suppose $(\boldsymbol{z},\boldsymbol{z}')$ are such that EQ holds. Also, suppose $(\boldsymbol{z},\boldsymbol{z}')$ are such that (3.4) holds, then we have $R_{11}(\boldsymbol{z}) \supset R_{11}(\boldsymbol{z}')$ and $R_{00}(\boldsymbol{z}) \subset R_{00}(\boldsymbol{z}')$, respectively, by Theorem 3.1. Define

$$\Delta(\boldsymbol{z},\boldsymbol{z}') \equiv \{R^*_{10}(\boldsymbol{z}) \cup R^*_{01}(\boldsymbol{z})\} \setminus \boldsymbol{R}_1(\boldsymbol{z}'), \tag{1.3}$$

$$-\Delta(\boldsymbol{z},\boldsymbol{z}') \equiv \left\{R^*_{10}(\boldsymbol{z}') \cup R^*_{01}(\boldsymbol{z}')\right\} \setminus \boldsymbol{R}_1(\boldsymbol{z}). \tag{1.4}$$

Consider partitions $\Delta(\boldsymbol{z},\boldsymbol{z}') = \Delta^1(\boldsymbol{z},\boldsymbol{z}') \cup \Delta^2(\boldsymbol{z},\boldsymbol{z}')$ and $-\Delta(\boldsymbol{z},\boldsymbol{z}') = -\Delta^1(\boldsymbol{z},\boldsymbol{z}') \cup -\Delta^2(\boldsymbol{z},\boldsymbol{z}')$ such that

$$\Delta^1(\boldsymbol{z},\boldsymbol{z}') \equiv R^*_{10}(\boldsymbol{z})\setminus\boldsymbol{R}_1(\boldsymbol{z}'), \quad \Delta^2(\boldsymbol{z},\boldsymbol{z}') \equiv R^*_{01}(\boldsymbol{z})\setminus\boldsymbol{R}_1(\boldsymbol{z}'),$$

$$-\Delta^1(\boldsymbol{z},\boldsymbol{z}') \equiv R^*_{10}(\boldsymbol{z}')\setminus\boldsymbol{R}_1(\boldsymbol{z}), \quad -\Delta^2(\boldsymbol{z},\boldsymbol{z}') \equiv R^*_{01}(\boldsymbol{z}')\setminus\boldsymbol{R}_1(\boldsymbol{z}).$$

That is, $\Delta^1(\boldsymbol{z},\boldsymbol{z}')$ and $-\Delta^1(\boldsymbol{z},\boldsymbol{z}')$ are regions of R^*_{10} exchanged with the regions for $\boldsymbol{D}=(0,0)$ and $\boldsymbol{D}=(1,1)$, respectively, and $+\Delta^2(\boldsymbol{z},\boldsymbol{z}')$ and $-\Delta^2(\boldsymbol{z},\boldsymbol{z}')$ are for R^*_{01}.

Before proceeding, we introduce the following general rule that is useful later: for a uniform random vector $\tilde{\boldsymbol{U}}$ and two sets B and B' contained in $\tilde{\mathcal{U}}$ and for a r.v. ϵ and set $A \subset \mathcal{E}$,

$$\Pr[\epsilon \in A, \tilde{\boldsymbol{U}} \in B] - \Pr[\epsilon \in A, \tilde{\boldsymbol{U}} \in B'] = \Pr[\epsilon \in A, \tilde{\boldsymbol{U}} \in B\setminus B'] - \Pr[\epsilon \in A, \tilde{\boldsymbol{U}} \in B'\setminus B]. \tag{1.5}$$

Since we do not use the variation in X, we suppress it throughout. Let $\mu_{\boldsymbol{d}} \equiv \mu_0 + \mu_1 d_1 + \mu_2 d_2$ for simplicity. Now, by Assumption IN,

$$\begin{aligned} h_{10}(\boldsymbol{z},\boldsymbol{z}') =& \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in R^*_{10}(\boldsymbol{z})] - \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in R^*_{10}(\boldsymbol{z}')] \\ =& \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in R^*_{10}(\boldsymbol{z})\backslash R^*_{10}(\boldsymbol{z}')] - \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in R^*_{10}(\boldsymbol{z}')\backslash R^*_{10}(\boldsymbol{z})] \\ =& \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in \Delta^1(\boldsymbol{z},\boldsymbol{z}')] - \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in -\Delta^1(\boldsymbol{z},\boldsymbol{z}')] \end{aligned}$$

where the second equality is by (1.5) and the third equality is by the following derivation:

$$\begin{aligned} R^*_{10}(\boldsymbol{z})\backslash R^*_{10}(\boldsymbol{z}') &= \left[\left\{R^*_{10}(\boldsymbol{z}) \cap \boldsymbol{R}_1(\boldsymbol{z}')^c\right\} \backslash R^*_{10}(\boldsymbol{z}')\right] \cup \left[\left\{R^*_{10}(\boldsymbol{z}) \cap \boldsymbol{R}_1(\boldsymbol{z}')\right\} \backslash R^*_{10}(\boldsymbol{z}')\right] \\ &= \left[\left\{R^*_{10}(\boldsymbol{z}) \cap \boldsymbol{R}_1(\boldsymbol{z}')^c\right\}\right] \cup \left[\left\{R^*_{10}(\boldsymbol{z}') \cap \boldsymbol{R}_1(\boldsymbol{z})\right\} \backslash R^*_{10}(\boldsymbol{z}')\right] \\ &= \Delta^1(\boldsymbol{z},\boldsymbol{z}'), \end{aligned}$$

where the first equality is by the distributive law and $\mathcal{U} = \boldsymbol{R}_1(\boldsymbol{z}')^c \cup \boldsymbol{R}_1(\boldsymbol{z}')$, the second equality is by $\boldsymbol{R}_1(\boldsymbol{z}')^c = R^*_{10}(\boldsymbol{z}')^c \cap R^*_{01}(\boldsymbol{z}')^c$ (the first term) and by Assumption EQ (the second term), and the last equality is by the definition of $\Delta^1(\boldsymbol{z},\boldsymbol{z}')$ and $\{R^*_{10}(\boldsymbol{z}') \cap \boldsymbol{R}_1(\boldsymbol{z})\} \backslash R^*_{10}(\boldsymbol{z}')$ being empty. Analogously, one can show that $R^*_{10}(\boldsymbol{z}')\backslash R^*_{10}(\boldsymbol{z}) = -\Delta^1(\boldsymbol{z},\boldsymbol{z}')$ using Assumption EQ and the definition of $-\Delta^1(\boldsymbol{z},\boldsymbol{z}')$. Similarly,

$$\begin{aligned} h_{01}(\boldsymbol{z},\boldsymbol{z}') =& \Pr[\epsilon \le \mu_{01}, \boldsymbol{U} \in R^*_{01}(\boldsymbol{z})] - \Pr[\epsilon \le \mu_{01}, \boldsymbol{U} \in R^*_{01}(\boldsymbol{z}')] \\ =& \Pr[\epsilon \le \mu_{01}, \boldsymbol{U} \in R^*_{01}(\boldsymbol{z})\backslash R^*_{01}(\boldsymbol{z}')] - \Pr[\epsilon \le \mu_{01}, \boldsymbol{U} \in R^*_{01}(\boldsymbol{z}')\backslash R^*_{01}(\boldsymbol{z})] \\ =& \Pr[\epsilon \le \mu_{01}, \boldsymbol{U} \in \Delta^2(\boldsymbol{z},\boldsymbol{z}')] - \Pr[\epsilon \le \mu_{01}, \boldsymbol{U} \in -\Delta^2(\boldsymbol{z},\boldsymbol{z}')]. \end{aligned}$$

Also, by the definitions of the partitions,

$$\begin{aligned} h_{11}(\boldsymbol{z},\boldsymbol{z}') =& \Pr[\epsilon \le \mu_{11}, \boldsymbol{U} \in -\Delta(\boldsymbol{z},\boldsymbol{z}') \cup A^*] \\ =& \Pr[\epsilon \le \mu_{11}, \boldsymbol{U} \in -\Delta^1(\boldsymbol{z},\boldsymbol{z}')] + \Pr[\epsilon \le \mu_{11}, \boldsymbol{U} \in -\Delta^2(\boldsymbol{z},\boldsymbol{z}')] \\ & + \Pr[\epsilon \le \mu_{11}, \boldsymbol{U} \in A^*] \end{aligned}$$

since $-\Delta(\boldsymbol{z},\boldsymbol{z}')$ and A^* are disjoint, and

$$
\begin{aligned}
h_{00}(\boldsymbol{z},\boldsymbol{z}') =& -\Pr[\epsilon \le \mu_{00}, \boldsymbol{U} \in \Delta(\boldsymbol{z},\boldsymbol{z}') \cup A^*] \\
=& -\Pr[\epsilon \le \mu_{00}, \boldsymbol{U} \in \Delta^1(\boldsymbol{z},\boldsymbol{z}')] - \Pr[\epsilon \le \mu_{00}, \boldsymbol{U} \in \Delta^2(\boldsymbol{z},\boldsymbol{z}')] \\
& - \Pr[\epsilon \le \mu_{00}, \boldsymbol{U} \in A^*]
\end{aligned}
$$

since $\Delta(\boldsymbol{z},\boldsymbol{z}')$ and A^* are disjoint. Now combining all the terms yields

$$
\begin{aligned}
h(\boldsymbol{z},\boldsymbol{z}') =& \Pr[\epsilon \le \mu_{11}, \boldsymbol{U} \in -\Delta^1(\boldsymbol{z},\boldsymbol{z}')] - \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in -\Delta^1(\boldsymbol{z},\boldsymbol{z}')] \\
& + \Pr[\epsilon \le \mu_{11}, \boldsymbol{U} \in -\Delta^2(\boldsymbol{z},\boldsymbol{z}')] - \Pr[\epsilon \le \mu_{01}, \boldsymbol{U} \in -\Delta^2(\boldsymbol{z},\boldsymbol{z}')] \\
& + \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in \Delta^1(\boldsymbol{z},\boldsymbol{z}')] - \Pr[\epsilon \le \mu_{00}, \boldsymbol{U} \in \Delta^1(\boldsymbol{z},\boldsymbol{z}')] \\
& + \Pr[\epsilon \le \mu_{01}, \boldsymbol{U} \in \Delta^2(\boldsymbol{z},\boldsymbol{z}')] - \Pr[\epsilon \le \mu_{00}, \boldsymbol{U} \in \Delta^2(\boldsymbol{z},\boldsymbol{z}')]. \\
& + \Pr[\epsilon \le \mu_{11}, \boldsymbol{U} \in A^*] - \Pr[\epsilon \le \mu_{00}, \boldsymbol{U} \in A^*]
\end{aligned}
$$

In this expression, each set of $\boldsymbol{U}$ has a corresponding set in the expression (2.12) of the main text: $-\Delta^1(\boldsymbol{z},\boldsymbol{z}') = \Delta_a$, $-\Delta^2(\boldsymbol{z},\boldsymbol{z}') = \Delta_b$, $\Delta^1(\boldsymbol{z},\boldsymbol{z}') = \Delta_c$, $\Delta^2(\boldsymbol{z},\boldsymbol{z}') = \Delta_d$, and $A^* = \Delta_e$. Then, as already argued in the text, $\mu_{1,\boldsymbol{d}_{-s}} - \mu_{0,\boldsymbol{d}_{-s}}$ share the same signs for all s and $\forall \boldsymbol{d}_{-s} \in \{0,1\}$ and therefore $sgn\{h(\boldsymbol{z},\boldsymbol{z}')\} = sgn\left\{\mu_{1,\boldsymbol{d}_{-s}} - \mu_{0,\boldsymbol{d}_{-s}}\right\}$.

1.4 Proof of Theorem 3.2

To reduce the notation, we suppress the conditioning of $X = x$ throughout the proof. For a set $\tilde{\mathcal{D}} \subset \mathcal{D}$, let $\tilde{p}_{\tilde{\mathcal{D}}}(\boldsymbol{z}) \equiv \Pr[Y = 1, \boldsymbol{D} \in \tilde{\mathcal{D}} | \boldsymbol{Z} = \boldsymbol{z}]$ and $p_{\tilde{\mathcal{D}}}(\boldsymbol{z}) \equiv \Pr[\boldsymbol{D} \in \tilde{\mathcal{D}} | \boldsymbol{Z} = \boldsymbol{z}]$. Then the bounds (3.10) and (3.11) can be rewritten as

$$
U_{\boldsymbol{d}^j} = \inf_{\boldsymbol{z}\in\mathcal{Z}} \left\{ \tilde{p}_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}) + p_{\mathcal{D}\setminus\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}) \right\}, \qquad L_{\boldsymbol{d}^j} = \sup_{\boldsymbol{z}\in\mathcal{Z}} \tilde{p}_{\mathcal{D}^{\leq}(\boldsymbol{d}^j)}(\boldsymbol{z}).
$$

Note that $\tilde{p}_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}) = \Pr[Y = 1 | \boldsymbol{Z} = \boldsymbol{z}] - \tilde{p}_{\mathcal{D}\setminus\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z})$. Suppose $\boldsymbol{z}, \boldsymbol{z}'$ are chosen such that $p_{\boldsymbol{d}}(\boldsymbol{z}) - p_{\boldsymbol{d}}(\boldsymbol{z}') = \Pr[\boldsymbol{U} \in \Delta_{\boldsymbol{d}}(\boldsymbol{z},\boldsymbol{z}')] - \Pr[\boldsymbol{U} \in -\Delta_{\boldsymbol{d}}(\boldsymbol{z},\boldsymbol{z}')] > 0 \ \forall \mathcal{D}^{\geq}(\boldsymbol{d}^j)$, where $\Delta_{\boldsymbol{d}}$ and $-\Delta_{\boldsymbol{d}}$ are defined in (1.17) and (1.18) below. Observe that each term in $U_{\boldsymbol{d}^j}$ satisfies

$$\tilde{p}_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}) - \tilde{p}_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}') = \sum_{\boldsymbol{d}\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)} \left(\Pr[\epsilon \leq \mu_{\boldsymbol{d}}, \boldsymbol{U} \in \Delta_{\boldsymbol{d}}(\boldsymbol{z},\boldsymbol{z}')] - \Pr[\epsilon \leq \mu_{\boldsymbol{d}}, \boldsymbol{U} \in -\Delta_{\boldsymbol{d}}(\boldsymbol{z},\boldsymbol{z}')]\right),$$

$$\begin{aligned} p_{\mathcal{D}\setminus\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}) - p_{\mathcal{D}\setminus\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}') &= -(p_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}) - p_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}')) \\ &= -\left(\sum_{\boldsymbol{d}\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)} \Pr[\boldsymbol{U} \in \Delta_{\boldsymbol{d}}(\boldsymbol{z},\boldsymbol{z}')] - \Pr[\boldsymbol{U} \in -\Delta_{\boldsymbol{d}}(\boldsymbol{z},\boldsymbol{z}')]\right), \end{aligned}$$

and thus

$$\begin{aligned} &\tilde{p}_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}) + p_{\mathcal{D}\setminus\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}) - \left\{\tilde{p}_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}') + p_{\mathcal{D}\setminus\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}')\right\} \\ &= -\sum_{\boldsymbol{d}\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)} \left(\Pr[\epsilon > \mu_{\boldsymbol{d}}, \boldsymbol{U} \in \Delta_{\boldsymbol{d}}(\boldsymbol{z},\boldsymbol{z}')] - \Pr[\epsilon > \mu_{\boldsymbol{d}}, \boldsymbol{U} \in -\Delta_{\boldsymbol{d}}(\boldsymbol{z},\boldsymbol{z}')]\right) < 0. \end{aligned}$$

Then this relationship creates a *partial ordering* of $\tilde{p}_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}) + p_{\mathcal{D}\setminus\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z})$ as a function of $\boldsymbol{z}$. According to this ordering, $\tilde{p}_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}) + p_{\mathcal{D}\setminus\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z})$ takes its smallest value as $\max_{\boldsymbol{d}(\boldsymbol{z})\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)} p_{\boldsymbol{d}(\boldsymbol{z})}(\boldsymbol{z})$ takes its largest value. Therefore, by (3.12),

$$U_{\boldsymbol{d}^j} = \inf_{\boldsymbol{z}\in\mathcal{Z}} \left\{\tilde{p}_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}) + p_{\mathcal{D}\setminus\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z})\right\} = \tilde{p}_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\bar{\boldsymbol{z}}) + p_{\mathcal{D}\setminus\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\bar{\boldsymbol{z}}).$$

By a similar argument, $L_{\boldsymbol{d}^j} = \sup_{\boldsymbol{z}\in\mathcal{Z}} \tilde{p}_{\mathcal{D}^{\leq}(\boldsymbol{d}^j)}(\boldsymbol{z}) = \tilde{p}_{\mathcal{D}^{\leq}(\boldsymbol{d}^j)}(\underline{\boldsymbol{z}})$.

To prove that these bounds on $E[Y_{\boldsymbol{d}^j}]$ are sharp, it suffices to show that for $s_j \in [L_{\boldsymbol{d}^j}, U_{\boldsymbol{d}^j}]$, there exists a density function $f^*_{\epsilon,\boldsymbol{U}}$ such that the following claims hold:

(A) $f^*_{\epsilon|\boldsymbol{U}}$ is strictly positive on $\mathbb{R}$.

(B) The proposed model is consistent with the data: $\forall \boldsymbol{d}$,

$$\Pr[\boldsymbol{D} = \boldsymbol{d}|\boldsymbol{Z} = \boldsymbol{z}] = \Pr[\boldsymbol{U}^* \in \boldsymbol{R}_{\boldsymbol{d}}(\boldsymbol{z})],$$

$$\Pr[Y = 1|\boldsymbol{D} = \boldsymbol{d}, \boldsymbol{Z} = \boldsymbol{z}] = \Pr[\epsilon^* \leq \mu_{\boldsymbol{d}}|\boldsymbol{U}^* \in \boldsymbol{R}_{\boldsymbol{d}}(\boldsymbol{z})],$$

(C) The proposed model is consistent with the specified values of $E[Y_{\boldsymbol{d}^j}]$: $\Pr[\epsilon^* \leq \mu_{\boldsymbol{d}^j}] = s_j$.

An argument similar to the proof of Theorem 3.1 and the partial ordering above establish the monotonicity of the event $\boldsymbol{U} \in \bigcup_{\boldsymbol{d}\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)} \boldsymbol{R}_{\boldsymbol{d}}(\boldsymbol{z})$ (and $\boldsymbol{U} \in \bigcup_{\boldsymbol{d}\in\mathcal{D}^{\leq}(\boldsymbol{d}^j)} \boldsymbol{R}_{\boldsymbol{d}}(\boldsymbol{z})$) w.r.t. $\boldsymbol{z}$. For example, for $\boldsymbol{z}, \boldsymbol{z}'$ chosen above, we have that $p_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}) - p_{\mathcal{D}^{\geq}(\boldsymbol{d}^j)}(\boldsymbol{z}') > 0$, and thus $\bigcup_{\boldsymbol{d}\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)} \boldsymbol{R}_{\boldsymbol{d}}(\boldsymbol{z}) \supset$

$\bigcup_{\boldsymbol{d}\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)} \boldsymbol{R_d}(\boldsymbol{z}')$, which implies

$$1\left[\boldsymbol{U}\in\bigcup_{\boldsymbol{d}\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)}\boldsymbol{R_d}(\boldsymbol{z})\right]-1\left[\boldsymbol{U}\in\bigcup_{\boldsymbol{d}\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)}\boldsymbol{R_d}(\boldsymbol{z}')\right]=1\left[\boldsymbol{U}\in\bigcup_{\boldsymbol{d}\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)}\boldsymbol{R_d}(\boldsymbol{z})\backslash\bigcup_{\boldsymbol{d}\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)}\boldsymbol{R_d}(\boldsymbol{z})\right]. \tag{1.6}$$

Given $1[\boldsymbol{D}\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)] = 1[\boldsymbol{U}\in\bigcup_{\boldsymbol{d}\in\mathcal{D}^{\geq}(\boldsymbol{d}^j)}\boldsymbol{R_d}(\boldsymbol{Z})]$, (1.6) is analogous to a scalar treatment decision $\tilde{D}=1[\tilde{D}=1]=1[\tilde{U}\leq\tilde{P}]$ with a scalar instrument $\tilde{P}$, where $1[\tilde{U}\leq p']-1[\tilde{U}\leq p]=1[p\leq\tilde{U}\leq p']$ for $p'>p$. Based on this result and the results for the first part of Theorem 3.2, we can modify the proof of Theorem 2.1(iii) in Shaikh and Vytlacil (2011) to show (A)–(C).

1.5 Proof of Lemma 3.2

We introduce a lemma that establishes the connection between Theorem 3.1 and Lemma 3.2.

Lemma 1.2. *Based on the results in Theorem 3.1,* $\tilde{h}(\boldsymbol{z},\boldsymbol{z}',\tilde{\boldsymbol{x}})\equiv\sum_{j=0}^{S}h_j(\boldsymbol{z},\boldsymbol{z}',x_j)$ *satisfies*

$$\tilde{h}(\boldsymbol{z},\boldsymbol{z}',\tilde{\boldsymbol{x}})=\sum_{j=1}^{S}\sum_{(1,\boldsymbol{d}_{-s})\in\mathcal{D}^j}\int_{\Delta_{(1,\boldsymbol{d}_{-s}),(0,\boldsymbol{d}_{-s})}}\{\vartheta((1,\boldsymbol{d}_{-s}),x_j;\boldsymbol{u})-\vartheta((0,\boldsymbol{d}_{-s}),x_{j-1};\boldsymbol{u})\}d\boldsymbol{u}, \tag{1.7}$$

where $\Delta_{\boldsymbol{d},\tilde{\boldsymbol{d}}}=\Delta_{\boldsymbol{d},\tilde{\boldsymbol{d}}}(\boldsymbol{z},\boldsymbol{z}')$ *is a partition of* $\Delta_{\boldsymbol{d}}(\boldsymbol{z},\boldsymbol{z}')$ *defined below.*

As a special case of this lemma, $\tilde{h}(\boldsymbol{z}',\boldsymbol{z},x,...,x)=h(\boldsymbol{z}',\boldsymbol{z},x)$ can be expressed as

$$h(\boldsymbol{z}',\boldsymbol{z},x)=\sum_{\boldsymbol{d}_{-s}}\int_{\Delta_{(1,\boldsymbol{d}_{-s}),(0,\boldsymbol{d}_{-s})}}\{\vartheta((1,\boldsymbol{d}_{-s}),x;\boldsymbol{u})-\vartheta((0,\boldsymbol{d}_{-s}),x;\boldsymbol{u})\}d\boldsymbol{u}. \tag{1.8}$$

We prove Lemma 1.2 by drawing on the result of Theorem 3.1. We first establish the relationship between $(\boldsymbol{R}_j(\boldsymbol{z}),\boldsymbol{R}_j(\boldsymbol{z}'))$ and $(\boldsymbol{R}_{j-1}(\boldsymbol{z}),\boldsymbol{R}_{j-1}(\boldsymbol{z}'))$, and then establish refined results for individual equilibrium regions. By Theorem 3.1, for $\boldsymbol{z}$ and $\boldsymbol{z}'$ such that (3.4) holds, we have

$$\boldsymbol{R}^j(\boldsymbol{z})\subseteq\boldsymbol{R}^j(\boldsymbol{z}') \tag{1.9}$$

for $j=0,...,S$, including $\boldsymbol{R}^S(\boldsymbol{z})=\boldsymbol{R}^S(\boldsymbol{z}')=\mathcal{U}$ as a trivial case. For those $\boldsymbol{z}$ and $\boldsymbol{z}'$, introduce notation

$$\Delta_j(\boldsymbol{z}, \boldsymbol{z}') \equiv \boldsymbol{R}_j(\boldsymbol{z}) \backslash \boldsymbol{R}_j(\boldsymbol{z}'), \tag{1.10}$$

$$-\Delta_j(\boldsymbol{z}, \boldsymbol{z}') \equiv \boldsymbol{R}_j(\boldsymbol{z}') \backslash \boldsymbol{R}_j(\boldsymbol{z}), \tag{1.11}$$

and

$$\Delta^j(\boldsymbol{z}, \boldsymbol{z}') \equiv \boldsymbol{R}^j(\boldsymbol{z}) \backslash \boldsymbol{R}^j(\boldsymbol{z}'). \tag{1.12}$$

Note that, for $j = 1, ..., S$,

$$\boldsymbol{R}_j(\cdot) = \boldsymbol{R}^j(\cdot) \backslash \boldsymbol{R}^{j-1}(\cdot), \tag{1.13}$$

since $\boldsymbol{R}^j(\boldsymbol{z}) \equiv \bigcup_{k=0}^{j} \boldsymbol{R}_k(\boldsymbol{z})$. Fix $j = 1, ..., S$. Consider

$$\begin{aligned}
\Delta_j(\boldsymbol{z}, \boldsymbol{z}') &= \left(\boldsymbol{R}^j(\boldsymbol{z}) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c\right) \cap \left(\boldsymbol{R}^j(\boldsymbol{z}') \cap \boldsymbol{R}^{j-1}(\boldsymbol{z}')^c\right)^c \\
&= \left(\boldsymbol{R}^j(\boldsymbol{z}) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c\right) \cap \left(\boldsymbol{R}^j(\boldsymbol{z}')^c \cup \boldsymbol{R}^{j-1}(\boldsymbol{z}')\right) \\
&= \left(\boldsymbol{R}^j(\boldsymbol{z}) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c \cap \boldsymbol{R}^j(\boldsymbol{z}')^c\right) \cup \left(\boldsymbol{R}^j(\boldsymbol{z}) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c \cap \boldsymbol{R}^{j-1}(\boldsymbol{z}')\right) \\
&= \left\{\left(\boldsymbol{R}^j(\boldsymbol{z}) \backslash \boldsymbol{R}^j(\boldsymbol{z}')\right) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c\right\} \cup \left\{\left(\boldsymbol{R}^{j-1}(\boldsymbol{z}') \backslash \boldsymbol{R}^{j-1}(\boldsymbol{z})\right) \cap \boldsymbol{R}^j(\boldsymbol{z})\right\} \\
&= \Delta^{j-1}(\boldsymbol{z}', \boldsymbol{z}) \cap \boldsymbol{R}^j(\boldsymbol{z}),
\end{aligned}$$

where the first equality is by plugging in (1.13) into (1.10), the third equality is by the distributive law, and the last equality is by (1.9) and hence $\left(\boldsymbol{R}^j(\boldsymbol{z}) \backslash \boldsymbol{R}^j(\boldsymbol{z}')\right) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c = \emptyset$. But

$$\Delta^{j-1}(\boldsymbol{z}', \boldsymbol{z}) \backslash \boldsymbol{R}^j(\boldsymbol{z}) = \Delta^{j-1}(\boldsymbol{z}', \boldsymbol{z}) \backslash \left(\Delta^{j-1}(\boldsymbol{z}', \boldsymbol{z}) \cap \boldsymbol{R}^j(\boldsymbol{z})\right).$$

Symmetrically, by changing the role of $\boldsymbol{z}$ and $\boldsymbol{z}'$, consider

$$\begin{aligned}
-\Delta_j(\boldsymbol{z}, \boldsymbol{z}') &= \left(\boldsymbol{R}^j(\boldsymbol{z}') \cap \boldsymbol{R}^{j-1}(\boldsymbol{z}')^c\right) \cap \left(\boldsymbol{R}^j(\boldsymbol{z}) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c\right)^c \\
&= \left\{\left(\boldsymbol{R}^j(\boldsymbol{z}') \backslash \boldsymbol{R}^j(\boldsymbol{z})\right) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z}')^c\right\} \cup \left\{\left(\boldsymbol{R}^{j-1}(\boldsymbol{z}) \backslash \boldsymbol{R}^{j-1}(\boldsymbol{z}')\right) \cap \boldsymbol{R}^j(\boldsymbol{z}')\right\} \\
&= \Delta^j(\boldsymbol{z}', \boldsymbol{z}) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z}')^c,
\end{aligned}$$

where the last equality is by (1.9) that $\boldsymbol{R}^{j-1}(\boldsymbol{z}) \subset \boldsymbol{R}^{j-1}(\boldsymbol{z}')$. But

$$\Delta^j(\boldsymbol{z}', \boldsymbol{z}) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z}')^c = \Delta^j(\boldsymbol{z}', \boldsymbol{z}) \backslash \left(\Delta^j(\boldsymbol{z}', \boldsymbol{z}) \backslash \boldsymbol{R}^{j-1}(\boldsymbol{z}')\right).$$

Note that

$$\Delta^{j-1}(\boldsymbol{z}', \boldsymbol{z}) \backslash \boldsymbol{R}^j(\boldsymbol{z}) = \Delta^j(\boldsymbol{z}', \boldsymbol{z}) \backslash \boldsymbol{R}^{j-1}(\boldsymbol{z}') \equiv A^*, \tag{1.14}$$

because

$$\begin{aligned}\Delta^{j-1}(\boldsymbol{z}', \boldsymbol{z}) \backslash \boldsymbol{R}^j(\boldsymbol{z}) &= \boldsymbol{R}^{j-1}(\boldsymbol{z}') \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c \cap \boldsymbol{R}^j(\boldsymbol{z})^c = \boldsymbol{R}^{j-1}(\boldsymbol{z}') \cap \boldsymbol{R}^j(\boldsymbol{z})^c \\ &= \boldsymbol{R}^j(\boldsymbol{z}') \cap \boldsymbol{R}^j(\boldsymbol{z})^c \cap \boldsymbol{R}^{j-1}(\boldsymbol{z}') = \Delta^j(\boldsymbol{z}', \boldsymbol{z}) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z}'),\end{aligned}$$

where the second equality is by $\boldsymbol{R}^{j-1}(\boldsymbol{z}) \subset \boldsymbol{R}^j(\boldsymbol{z})$ and the third equality is by $\boldsymbol{R}^{j-1}(\boldsymbol{z}') \subset \boldsymbol{R}^j(\boldsymbol{z}')$. In sum,

$$\Delta_j(\boldsymbol{z}, \boldsymbol{z}') = \Delta^{j-1}(\boldsymbol{z}', \boldsymbol{z}) \backslash A^*, \qquad -\Delta_j(\boldsymbol{z}, \boldsymbol{z}') = \Delta^j(\boldsymbol{z}', \boldsymbol{z}) \backslash A^*. \tag{1.15}$$

(1.15) shows how the outflow ($\Delta_j(\boldsymbol{z}, \boldsymbol{z}')$) and inflow ($-\Delta_j(\boldsymbol{z}, \boldsymbol{z}')$) of $\boldsymbol{R}_j$ can be written in terms of the inflows of $\boldsymbol{R}^{j-1}$ and $\boldsymbol{R}^j$, respectively. And figuratively, A^* adjusts for the "leakage" when the change from $\boldsymbol{z}$ to $\boldsymbol{z}'$ is relatively large. Therefore, by (1.15), we have the inflow and outflow match result between $\boldsymbol{R}_j$ and $\boldsymbol{R}_{j-1}$:

$$\Delta_j(\boldsymbol{z}, \boldsymbol{z}') = -\Delta_{j-1}(\boldsymbol{z}, \boldsymbol{z}') \tag{1.16}$$

Now, we want to decompose this match into matches of flows in individual $\boldsymbol{R}_{\boldsymbol{d}^j}$'s. Define

$$\Delta_{\boldsymbol{d}^j}(\boldsymbol{z}, \boldsymbol{z}') \equiv \boldsymbol{R}^*_{\boldsymbol{d}^j}(\boldsymbol{z}) \backslash \boldsymbol{R}^*_{\boldsymbol{d}^j}(\boldsymbol{z}'), \tag{1.17}$$

$$-\Delta_{\boldsymbol{d}^j}(\boldsymbol{z}, \boldsymbol{z}') \equiv \boldsymbol{R}^*_{\boldsymbol{d}^j}(\boldsymbol{z}') \backslash \boldsymbol{R}^*_{\boldsymbol{d}^j}(\boldsymbol{z}). \tag{1.18}$$

By Assumption EQ (or (1.1)),

$$\Delta_{\boldsymbol{d}^j}(\boldsymbol{z},\boldsymbol{z}') = \boldsymbol{R}^*_{\boldsymbol{d}^j}(\boldsymbol{z})\backslash\boldsymbol{R}_j(\boldsymbol{z}'),$$
$$-\Delta_{\boldsymbol{d}^j}(\boldsymbol{z},\boldsymbol{z}') = \boldsymbol{R}^*_{\boldsymbol{d}^j}(\boldsymbol{z}')\backslash\boldsymbol{R}_j(\boldsymbol{z}),$$

and therefore,

$$\Delta_j(\boldsymbol{z},\boldsymbol{z}') = \bigcup_{\boldsymbol{d}^j}\Delta_{\boldsymbol{d}^j}(\boldsymbol{z},\boldsymbol{z}'),$$
$$-\Delta_j(\boldsymbol{z},\boldsymbol{z}') = \bigcup_{\boldsymbol{d}^j}-\Delta_{\boldsymbol{d}^j}(\boldsymbol{z},\boldsymbol{z}'),$$

since $\boldsymbol{R}_j(\cdot) = \bigcup_{\boldsymbol{d}^j}\boldsymbol{R}^*_{\boldsymbol{d}^j}(\cdot)$. Also, note that $\{\Delta_{\boldsymbol{d}^j}(\boldsymbol{z},\boldsymbol{z}')\}_{\boldsymbol{d}^j}$ are disjoint since $\{\boldsymbol{R}^*_{\boldsymbol{d}^j}(\boldsymbol{z})\}_{\boldsymbol{d}^j}$ are disjoint. Therefore, $\{\Delta_{\boldsymbol{d}^j}(\boldsymbol{z},\boldsymbol{z}')\}_{\boldsymbol{d}^j}$ and $\{-\Delta_{\boldsymbol{d}^j}(\boldsymbol{z},\boldsymbol{z}')\}_{\boldsymbol{d}^j}$ are partitions of $\Delta_j(\boldsymbol{z},\boldsymbol{z}')$ and $-\Delta_j(\boldsymbol{z},\boldsymbol{z}')$, respectively. Then, suppressing $(\boldsymbol{z},\boldsymbol{z}')$, rewrite (1.16) as

$$\bigcup_{\boldsymbol{d}^j}\Delta_{\boldsymbol{d}^j} = \bigcup_{\boldsymbol{d}^{j-1}}-\Delta_{\boldsymbol{d}^{j-1}}.$$

Note that, for any $\boldsymbol{d}^j$ and $\boldsymbol{d}^{j-1}$, $\Delta_{\boldsymbol{d}^j}$ does not necessarily coincide with $-\Delta_{\boldsymbol{d}^{j-1}}$. Therefore, we proceed as follows. For a given $\bar{\boldsymbol{d}}^j$, we further partition $\Delta_{\bar{\boldsymbol{d}}^j}$ by considering $\{\Delta_{\bar{\boldsymbol{d}}^j,\boldsymbol{d}^{j-1}}\}_{\boldsymbol{d}^{j-1}}$ with $\Delta_{\bar{\boldsymbol{d}}^j,\boldsymbol{d}^{j-1}} = \emptyset$ for $\boldsymbol{d}^j \neq \bar{\boldsymbol{d}}^j$ and $\boldsymbol{d}^j \in \mathcal{D}^>(\boldsymbol{d}^{j-1})$. Likewise, for a given $\bar{\boldsymbol{d}}^{j-1}$, partition $-\Delta_{\bar{\boldsymbol{d}}^{j-1}}$ by considering $\{-\Delta_{\bar{\boldsymbol{d}}^{j-1},\boldsymbol{d}^j}\}_{\boldsymbol{d}^j}$ with $-\Delta_{\bar{\boldsymbol{d}}^{j-1},\boldsymbol{d}^j} = \emptyset$ for $\boldsymbol{d}^{j-1} \neq \bar{\boldsymbol{d}}^{j-1}$ and $\boldsymbol{d}^{j-1} \in \mathcal{D}^<(\boldsymbol{d}^j)$. Then,

$$\Delta_{\boldsymbol{d}^j} = \bigcup_{\boldsymbol{d}^{j-1}}\Delta_{\boldsymbol{d}^j,\boldsymbol{d}^{j-1}}, \tag{1.19}$$

$$-\Delta_{\boldsymbol{d}^{j-1}} = \bigcup_{\boldsymbol{d}^j}-\Delta_{\boldsymbol{d}^{j-1},\boldsymbol{d}^j}, \tag{1.20}$$

with

$$\Delta_{\boldsymbol{d}^j,\boldsymbol{d}^{j-1}} = -\Delta_{\boldsymbol{d}^{j-1},\boldsymbol{d}^j}. \tag{1.21}$$

Now, for a given $\boldsymbol{d}^j$ and $j = 1, ..., S-1$,

$$\begin{aligned}
& h_{\boldsymbol{d}^j}(\boldsymbol{z}, \boldsymbol{z}', x) \\
&= \int_{\boldsymbol{R}^*_{\boldsymbol{d}^j}(\boldsymbol{z})} \vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) d\boldsymbol{u} - \int_{\boldsymbol{R}^*_{\boldsymbol{d}^j}(\boldsymbol{z}')} \vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) d\boldsymbol{u} \\
&= \int_{\Delta_{\boldsymbol{d}^j}} \vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) d\boldsymbol{u} - \int_{-\Delta_{\boldsymbol{d}^j}} \vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) d\boldsymbol{u} \\
&= \sum_{\boldsymbol{d}^{j-1}} \int_{\Delta_{\boldsymbol{d}^j, \boldsymbol{d}^{j-1}}} \vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) d\boldsymbol{u} - \sum_{\boldsymbol{d}^{j+1}} \int_{-\Delta_{\boldsymbol{d}^j, \boldsymbol{d}^{j+1}}} \vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) d\boldsymbol{u} \\
&= \sum_{\boldsymbol{d}^{j-1}} \int_{\Delta_{\boldsymbol{d}^j, \boldsymbol{d}^{j-1}}} \vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) d\boldsymbol{u} - \sum_{\boldsymbol{d}^{j+1}} \int_{\Delta_{\boldsymbol{d}^{j+1}, \boldsymbol{d}^j}} \vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) d\boldsymbol{u}, \qquad (1.22)
\end{aligned}$$

where the second equality is by (1.19)–(1.20), and the third equality is by (1.21). Also, for $j = 0$,

$$\begin{aligned}
& \int_{\boldsymbol{R}^*_{\boldsymbol{d}^0}(\boldsymbol{z})} \vartheta(\boldsymbol{d}^0, x; \boldsymbol{u}) d\boldsymbol{u} - \int_{\boldsymbol{R}^*_{\boldsymbol{d}^0}(\boldsymbol{z}')} \vartheta(\boldsymbol{d}^0, x; \boldsymbol{u}) d\boldsymbol{u} \\
&= -\sum_{\boldsymbol{d}^1} \int_{\Delta_{\boldsymbol{d}^1, \boldsymbol{d}^0}} \vartheta(\boldsymbol{d}^0, x; \boldsymbol{u}) d\boldsymbol{u}, \qquad (1.23)
\end{aligned}$$

since $\Delta_{\boldsymbol{d}^0}(\boldsymbol{z}, \boldsymbol{z}') = \emptyset$ by the choice of $(\boldsymbol{z}, \boldsymbol{z}')$. And, for $j = S$,

$$\begin{aligned}
& \int_{\boldsymbol{R}^*_{\boldsymbol{d}^S}(\boldsymbol{z})} \vartheta(\boldsymbol{d}^S, x; \boldsymbol{u}) d\boldsymbol{u} - \int_{\boldsymbol{R}^*_{\boldsymbol{d}^S}(\boldsymbol{z}')} \vartheta(\boldsymbol{d}^S, x; \boldsymbol{u}) d\boldsymbol{u} \\
&= \sum_{\boldsymbol{d}^{S-1}} \int_{\Delta_{\boldsymbol{d}^S, \boldsymbol{d}^{S-1}}} \vartheta(\boldsymbol{d}^S, x; \boldsymbol{u}) d\boldsymbol{u}, \qquad (1.24)
\end{aligned}$$

since $-\Delta_S(\boldsymbol{z}, \boldsymbol{z}') = \emptyset$ by the choice of $(\boldsymbol{z}, \boldsymbol{z}')$. Therefore, by combining (1.22)–(1.24), we have

$$\begin{aligned}
h(\boldsymbol{z}, \boldsymbol{z}', x) &= \sum_{j=0}^{S} \sum_{\boldsymbol{d}^j} \left\{ \int_{\boldsymbol{R}^*_{\boldsymbol{d}^j}(\boldsymbol{z})} \vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) d\boldsymbol{u} - \int_{\boldsymbol{R}^*_{\boldsymbol{d}^j}(\boldsymbol{z}')} \vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) d\boldsymbol{u} \right\} \\
&= \sum_{j=0}^{S} \sum_{\boldsymbol{d}^j} \left\{ \sum_{\boldsymbol{d}^{j-1}} \int_{\Delta_{\boldsymbol{d}^j, \boldsymbol{d}^{j-1}}} \vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) d\boldsymbol{u} - \sum_{\boldsymbol{d}^{j+1}} \int_{\Delta_{\boldsymbol{d}^{j+1}, \boldsymbol{d}^j}} \vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) d\boldsymbol{u} \right\} \\
&= \sum_{j=1}^{S} \sum_{\boldsymbol{d}^j} \sum_{\boldsymbol{d}^{j-1}} \int_{\Delta_{\boldsymbol{d}^j, \boldsymbol{d}^{j-1}}} \{\vartheta(\boldsymbol{d}^j, x; \boldsymbol{u}) - \vartheta(\boldsymbol{d}^{j-1}, x; \boldsymbol{u})\} d\boldsymbol{u} \\
&= \sum_{\boldsymbol{d}_{-s}} \int_{\Delta_{(1, \boldsymbol{d}_{-s}), (0, \boldsymbol{d}_{-s})}} \{\vartheta((1, \boldsymbol{d}_{-s}), x; \boldsymbol{u}) - \vartheta((0, \boldsymbol{d}_{-s}), x; \boldsymbol{u})\} d\boldsymbol{u},
\end{aligned}$$

where the last equality is by the definition of $\Delta_{\boldsymbol{d}^j, \boldsymbol{d}^{j-1}}$. Also, by a similar argument, we can show that

$$
\begin{aligned}
\tilde{h}(\boldsymbol{z}, \boldsymbol{z}', \tilde{\boldsymbol{x}}) &= \sum_{j=1}^{S} \sum_{\boldsymbol{d}^j} \sum_{\boldsymbol{d}^{j-1}} \int_{\Delta_{\boldsymbol{d}^j, \boldsymbol{d}^{j-1}}} \{\vartheta(\boldsymbol{d}^j, x_j; \boldsymbol{u}) - \vartheta(\boldsymbol{d}^{j-1}, x_j; \boldsymbol{u})\} d\boldsymbol{u} \\
&= \sum_{j=1}^{S} \sum_{(1, \boldsymbol{d}_{-s}) \in \mathcal{D}^j} \int_{\Delta_{(1, \boldsymbol{d}_{-s}), (0, \boldsymbol{d}_{-s})}} \{\vartheta((1, \boldsymbol{d}_{-s}), x_j; \boldsymbol{u}) - \vartheta((0, \boldsymbol{d}_{-s}), x_{j-1}; \boldsymbol{u})\} d\boldsymbol{u}. \qquad (1.25)
\end{aligned}
$$

This completes the proof of Lemma 1.2.

Now we prove Lemma 3.2. For part (i), suppose that $\vartheta(1, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) - \vartheta(0, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) > 0$ a.e. $\boldsymbol{u}$ $\forall \boldsymbol{d}_{-s}, s$. Then by (1.8), $h > 0$. Conversely, if $h > 0$ then it should be that $\vartheta(1, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) - \vartheta(0, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) > 0$ a.e. $\boldsymbol{u}$ $\forall \boldsymbol{d}_{-s}, s$. Suppose not and suppose $\vartheta(1, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) - \vartheta(0, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) \leq 0$ with positive measure for some $\boldsymbol{d}_{-s}$ and s. Then by Assumption M, this implies that $\vartheta(1, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) - \vartheta(0, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) \leq 0$ $\forall \boldsymbol{d}_{-s}, s$ a.e. $\boldsymbol{u}$, and thus $h \leq 0$ which is contradiction. By applying similar arguments for other signs, we have the desired result. Now we prove part (ii). Note that (1.25) can be rewritten as

$$
\begin{aligned}
&\tilde{h}(\boldsymbol{z}, \boldsymbol{z}', \tilde{\boldsymbol{x}}) - \sum_{k \neq j} \sum_{(1, \boldsymbol{d}_{-s}) \in \mathcal{D}^k} \int_{\Delta_{(1, \boldsymbol{d}_{-s}), (0, \boldsymbol{d}_{-s})}} \{\vartheta((1, \boldsymbol{d}_{-s}), x_k; \boldsymbol{u}) - \vartheta((0, \boldsymbol{d}_{-s}), x_{k-1}; \boldsymbol{u})\} \, d\boldsymbol{u} \\
&= \sum_{(1, \boldsymbol{d}_{-s}) \in \mathcal{D}^j} \int_{\Delta_{(1, \boldsymbol{d}_{-s}), (0, \boldsymbol{d}_{-s})}} \{\vartheta((1, \boldsymbol{d}_{-s}), x_j; \boldsymbol{u}) - \vartheta((0, \boldsymbol{d}_{-s}), x_{j-1}; \boldsymbol{u})\} \, d\boldsymbol{u}. \qquad (1.26)
\end{aligned}
$$

We prove the case $\iota = 1$; the proof for the other cases follows symmetrically. For $k \neq j$, when $-\vartheta((1, \boldsymbol{d}_{-s}), x_k; \boldsymbol{u}) + \vartheta((0, \boldsymbol{d}_{-s}), x_{k-1}; \boldsymbol{u}) > 0$ a.e. $\boldsymbol{u}$ $\forall (1, \boldsymbol{d}_{-s}) \in \mathcal{D}^k$, it satisfies

$$
- \sum_{(1, \boldsymbol{d}_{-s}) \in \mathcal{D}^k} \int_{\Delta_{(1, \boldsymbol{d}_{-s}), (0, \boldsymbol{d}_{-s})}} \{\vartheta((1, \boldsymbol{d}_{-s}), x_k; \boldsymbol{u}) - \vartheta((0, \boldsymbol{d}_{-s}), x_{k-1}; \boldsymbol{u})\} \, d\boldsymbol{u} > 0.
$$

Combining with $\tilde{h}(\boldsymbol{z}, \boldsymbol{z}', \tilde{\boldsymbol{x}}) > 0$ implies that the l.h.s. of (1.26) is positive. This implies that $\vartheta((1, \boldsymbol{d}_{-s}), x_j; \boldsymbol{u}) - \vartheta((0, \boldsymbol{d}_{-s}), x_{j-1}; \boldsymbol{u}) > 0$ a.e. $\boldsymbol{u}$ $\forall (1, \boldsymbol{d}_{-s}) \in \mathcal{D}^j$. If not, then it results in a contradiction as in the previous argument.

1.6 Proof of Theorem 3.3

Consider

$$E[Y_{\boldsymbol{d}^j}|X=x] = E[Y|\boldsymbol{D}=\boldsymbol{d}^j,\boldsymbol{Z}=\boldsymbol{z},X=x]\Pr[\boldsymbol{D}=\boldsymbol{d}^j|\boldsymbol{Z}=\boldsymbol{z}] + \sum_{\boldsymbol{d}'\neq\boldsymbol{d}^j} E[Y_{\boldsymbol{d}^j}|\boldsymbol{D}=\boldsymbol{d}',\boldsymbol{Z}=\boldsymbol{z},X=x]\Pr[\boldsymbol{D}=\boldsymbol{d}'|\boldsymbol{Z}=\boldsymbol{z}]. \tag{1.27}$$

Consider $j'<j$ for $E[Y_{\boldsymbol{d}^j}|\boldsymbol{D}=\boldsymbol{d}^{j'},\boldsymbol{Z},X]$ in (1.27) with $\boldsymbol{d}^{j'}\in\mathcal{D}^<(\boldsymbol{d}^j)$. Then, for example, if $(x_k,x_{k-1})\in\mathcal{X}_{k,k-1}(-1)\cup\mathcal{X}_{k,k-1}(0)$ for $j'+1\le k\le j$, then $\vartheta(\boldsymbol{d}^j,x;\boldsymbol{u})\le\vartheta(\boldsymbol{d}^{j'},x';\boldsymbol{u})$ where $x=x_j$ and $x'=x_{j'}$ by transitively applying (3.13). Therefore

$$\begin{aligned}
E[Y_{\boldsymbol{d}^j}|\boldsymbol{D}=\boldsymbol{d}^{j'},\boldsymbol{Z}=\boldsymbol{z},X=x] &= E[\theta(\boldsymbol{d}^j,x,\epsilon)|\boldsymbol{U}\in R_{\boldsymbol{d}^{j'}}(\boldsymbol{z}),\boldsymbol{Z}=\boldsymbol{z},X=x]\\
&= \frac{1}{\Pr[\boldsymbol{U}\in R_{\boldsymbol{d}^{j'}}(\boldsymbol{z})]}\int_{R_{\boldsymbol{d}^{j'}}(\boldsymbol{z})}\vartheta(\boldsymbol{d}^j,x;\boldsymbol{u})d\boldsymbol{u}\\
&\le \frac{1}{\Pr[\boldsymbol{U}\in R_{\boldsymbol{d}^{j'}}(\boldsymbol{z})]}\int_{R_{\boldsymbol{d}^{j'}}(\boldsymbol{z})}\vartheta(\boldsymbol{d}^{j'},x';\boldsymbol{u})d\boldsymbol{u}\\
&= E[\theta(\boldsymbol{d}^{j'},x',\epsilon)|\boldsymbol{U}\in R_{\boldsymbol{d}^{j'}}(\boldsymbol{z}),\boldsymbol{Z}=\boldsymbol{z},X=x']\\
&= E[Y|\boldsymbol{D}=\boldsymbol{d}^{j'},\boldsymbol{Z}=\boldsymbol{z},X=x'].
\end{aligned} \tag{1.28}$$

Symmetrically, for $j'>j$, if $(x_k,x_{k-1})\in\mathcal{X}_{k,k-1}(1)\cup\mathcal{X}_{k,k-1}(0)$ for $j+1\le k\le j'$, then $\vartheta(\boldsymbol{d}^j,x;\boldsymbol{u})\le\vartheta(\boldsymbol{d}^{j'},x';\boldsymbol{u})$ where $\boldsymbol{d}^{j'}\in\mathcal{D}^>(\boldsymbol{d}^j)$, $x=x_j$ and $x'=x_{j'}$. Therefore the same bound as (1.28) is derived. Given these results, to collect all $x'\in\mathcal{X}$ that yield $\vartheta(\boldsymbol{d}^j,x;\boldsymbol{u})\le\vartheta(\boldsymbol{d}^{j'},x';\boldsymbol{u})$ for $\boldsymbol{d}^{j'}\in\mathcal{D}^<(\boldsymbol{d}^j)\cup\mathcal{D}^>(\boldsymbol{d}^j)$, we can construct a set

$$\begin{aligned}
x'\in&\left\{x_{j'}:(x_k,x_{k-1})\in\mathcal{X}_{k,k-1}(-1)\cup\mathcal{X}_{k,k-1}(0)\text{ for }j'+1\le k\le j,x_j=x\right\}\\
&\cup\left\{x_{j'}:(x_k,x_{k-1})\in\mathcal{X}_{k,k-1}(1)\cup\mathcal{X}_{k,k-1}(0)\text{ for }j+1\le k\le j',x_j=x\right\}.
\end{aligned}$$

Then we can further shrink the bound in (1.28) by taking the infimum over all x' in this set. The lower bound on $E[Y_{\boldsymbol{d}^j}|\boldsymbol{D}=\boldsymbol{d}^{j'},\boldsymbol{Z}=\boldsymbol{z},X=x]$ can be constructed by simply choosing the opposite signs in the preceding argument. Since the other terms in (1.27) are observed, we have the desired bounds in the theorem.

2 Airline and Pollution Data

We combine data spanning the period 2000–2015 from two sources: airline data from the U.S. Department of Transportation and pollution data from the Environmental Protection Agency (EPA).

Airline Data. Our first data source contains airline information and combines publicly available data from the Department of Transportation's Origin and Destination Survey (DB1B) and Domestic Segment (T-100) database. These datasets have been used extensively in the literature to analyze the airline industry (see, e.g., Borenstein (1989), Berry (1992), Ciliberto and Tamer (2009), and more recently, Li et al. (2018) and Ciliberto et al. (2018)). The DB1B database is a quarterly sample of all passenger domestic itineraries. The dataset contains coupon-specific information, including origin and destination airports, number of coupons, the corresponding operating carriers, number of passengers, prorated market fare, market miles flown, and distance. The T-100 dataset is a monthly census of all domestic flights broken down by airline, and origin and destination airports.

Our time-unit of analysis is a quarter and we define a market as the market for air connection between a pair of airports (regardless of intermediate stops) in a given quarter.[2] We restrict the sample to include the top 100 metropolitan statistical areas (MSA's), ranked by population at the beginning of our sample period. We follow Berry (1992) and Ciliberto and Tamer (2009) and define an airline as actively serving a market in a given quarter, if we observe at least 90 passengers in the DB1B survey flying with the airline in the corresponding quarter.[3] We exclude from our sample city pairs in which no airline operates in the whole sample period. Notice that we do include markets that are temporarily not served by any airline. This leaves us with 181,095 market-quarter observations.

In our analysis, we allow for airlines to have a heterogeneous effect on pollution, and to simplify computation, in each market we allow for six potential participants: American (AA), Delta (DL), United (UA), Southwest (WN), a medium-size airline, and a low-cost carrier.[4] The latter is not a bad approximation to the data in that we rarely observe more than one medium-size or low-cost in a market but it assumes that all low-cost airlines have the same strategic behavior, and so do

[2] In cities that operate more than one airport, we assume that flights to different airports in the same metropolitan area are in separate markets.

[3] This corresponds to approximately the number of passengers that would be carried on a medium-size jet operating once a week.

[4] That is, to limit the number of potential market structures, we lump together all the low cost carriers into one category, and Northwest, Continental, America West, and USAir under the medium airline type.

Table 1: Distribution of the Number of Carriers by Market Size

	Market size			
# firms	Large	Medium	Small	Total
0	7.96	8.20	8.62	8.18
1	41.18	22.53	20.58	30.30
2	28.14	23.41	21.25	25.04
3	12.65	20.00	16.67	16.05
4	7.65	14.72	15.17	11.51
5	1.98	9.90	16.48	7.80
6+	0.52	1.23	2.21	1.12
# markets	79,326	64,191	37,578	181,095

the medium airlines. Table 1 shows the number of firms in each market broken down by size as measured by population. As the table shows, market size alone does not explain market structure, a point first made by Ciliberto and Tamer (2009).

In our application, we consider two instruments for the entry decisions. The first is the *airport presence* of an airline proposed by Berry (1992). For a given airline, this variable is constructed as the number of markets it serves out of an airport as a fraction of the total number of markets served by all airlines out of the airport. A hub-and-spoke network allows firms to exploit demand-side and cost-side economies, which should affect the firm's profitability. While Berry (1992) assumes that an airline's airport presence only affects its own profits (and hence, is excluded from rivals' profits), Ciliberto and Tamer (2009) argue that this may not be the case in practice, since airport presence might be a measure of product differentiation, rendering it likely to enter the profit function of all firms through demand. While an instrument that enters all of the profit functions is fine in our context (see Appendix C.4), we also consider the instrument proposed by Ciliberto and Tamer (2009), which captures shocks to the fixed cost of providing a service in a market. This variable, which they call *cost*, is constructed as the percentage of the nonstop distance that the airline must travel in excess of the nonstop distance, if the airline uses a connecting instead of a nonstop flight.[5] Arguably, this variable only affects its own profits and is excluded from rivals' profits.

Table 2 presents the summary statistics of the airline related variables. Of the leading airlines, we see that American and Delta are present in about half of the markets, while United and Southwest are only present in about a quarter of the markets. American and Delta tend to dominate the

[5] Mechanically, the variable is constructed as the difference between the sum of the distances of a market's endpoints and the closest hub of an airline, and the nonstop distance between the endpoints, divided by the nonstop distance.

Table 2: Airline Summary Statistics

		American	Delta	United	Southwest	medium	low-cost
Market presence (0/1)	mean	0.44	0.57	0.28	0.25	0.56	0.17
	sd	0.51	0.51	0.46	0.44	0.51	0.38
Airport presence (%)	mean	0.43	0.56	0.27	0.25	0.39	0.10
	sd	0.17	0.18	0.16	0.18	0.14	0.08
Cost (%)	mean	0.71	0.41	0.76	0.29	0.22	0.04
	sd	1.56	1.28	1.43	0.83	0.60	0.17

airports in which they operate more than United and Southwest. From the cost variable, we see that both American and United tend to operate a hub-and-spoke network, while Southwest (and to a lesser extent Delta) operates most markets nonstop.

Pollution Data. The second component of our dataset is the air pollution data. The EPA compiles a database of outdoor concentrations of pollutants measured at more than 4,000 monitoring stations throughout the U.S., owned and operated mainly by state environmental agencies. Each monitoring station is geocoded, and hence, we are able to merge these data with the airline dataset by matching all the monitoring stations that are located within a 10km radius of each airport in our first dataset.

The principal emissions of aircraft include the greenhouse gases carbon dioxide (CO_2) and water vapor (H_2O), which have a direct impact on climate change. Aircraft jet engines also produce nitric oxide (NO) and nitrogen dioxide (NO_2) (which together are termed nitrogen oxides (NO_x)), carbon monoxide (CO), oxides of sulphur (SO_x), unburned or partially combusted hydrocarbons (also known as volatile organic compounds or VOC's), particulates, and other trace compounds (see, Federal Aviation Administration (2015)). In addition, ozone (O_3) is formed by the reaction of VOC's and NO_x in the presence of heat and sunlight. The set of pollutants other than CO_2 are more pernicious in that they can harm human health directly and can result in respiratory, cardiovascular, and neurological conditions. Research to date indicates that fine particulate matter (PM) is responsible for the majority of the health risks from aviation emissions, although ozone has a substantial health impact too.[6] Therefore, as our measure of pollution, we will consider both.

Our measure of ozone is a quarterly mean of daily maximum levels in parts per million. In terms of PM, as a general rule, the smaller the particle the further it travels in the atmosphere, the longer it remains suspended in the atmosphere, and the more risk it poses to human health. PM that measure

[6]See Federal Aviation Administration (2015).

Table 3: Market-level Summary Statistics

	Mean	Std. Dev.
Pollution		
Ozone (O_3)	.0477	.0056
Particulate matter (PM2.5)	8.3881	2.5287
Other controls		
Market size (pop.)	2307187.8	1925533.4
Income (per capita)	34281.6	4185.5
# of markets	181,095	

less than 2.5 micrometer can be readily inhaled, and thus, potentially pose increased health risks. The variable PM2.5 is a quarterly average of daily averages and is measured in micrograms/cubic meter. For each airport in our sample, we take an average (weighted by distance to the airport) of the data from all air monitoring stations within a 10km radius. The top panel of Table 3 shows the summary statistics of the pollution measures.

Other Market-Level Controls. We also include in our analysis market-level covariates that may affect both market structure and pollution levels. In particular, we construct a measure of market size by computing the (geometric) mean of the MSA populations at the market endpoints and a measure of economic activity by computing the average per capita income at the market endpoints, using data from the Regional Economic Accounts of the Bureau of Economic Analysis.

Finally, as we mentioned in Section 3.4, having access to data on a variable that affects pollution but is excluded from the airline participation decisions can greatly help in calculating the bounds of the ATE. Therefore, we construct a variable that measures the economic activity of pollution related industries (manufacturing, construction, and transportation other than air transportation) in a given market (MSA) as a fraction of total economic activity in that market, again, using data from the Regional Economic Accounts of the Bureau of Economic Analysis.

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